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On the Exponential Integrability of Conjugate Functions

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Abstract
We relate the exponential integrability of the conjugate function $\tilde{f}$ to the size of the gap in the essential range of f. Our main result complements a related theorem of Zygmund.

Keywords Exponential integrability · Conjugate function · Hilbert transform · Outer functions

Mathematics Subject Classification 42A50

1 Introduction

We denote by L^p the usual Lebesgue spaces of functions on the unit circle $\mathbb{T}$ with norm $\|\cdot\|_p$. Given $f \in L^1$, let u be the Poisson integral of f and denote by $\tilde{u}$ the harmonic conjugate function of u, normalized so that $\tilde{u}(0) = 0$. Then $\tilde{u}(z)$ has nontangential limit $\tilde{f}(\theta)$ almost everywhere on $\mathbb{T}$ and we call $\tilde{f}$ the conjugate function of f. Alternatively,

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the conjugate function $\tilde{f}$ can be defined as the principal value integral

$$\tilde{f}(\theta) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{|\theta-\varphi|>\epsilon} \cot\left(\frac{\theta-\varphi}{2}\right) f(\varphi)\, d\varphi \tag{1.1}$$

for almost every θ. For further details and references, see Sect. 2.1 below.

The linear mapping $f \mapsto \tilde{f}$ is referred to as the conjugation operator. If f is a trigonometric polynomial $\sum_{n=-N}^{N} a_n e^{in\theta}$, then $\tilde{f}$ is a trigonometric polynomial of the same degree

$$\tilde{f}(\theta) = \sum_{n=-N}^{N} -i \operatorname{sgn}(n) a_n e^{in\theta},$$

where $-i$ sgn is the Fourier multiplier associated with the conjugation operator.

When $1 < p < \infty$, according to a famous theorem of M. Riesz, there is a constant C_p such that

$$\|\tilde{f}\|_p \le C_p \|f\|_p$$

for all $f \in L^p$. In addition, although $f \in L^\infty$ does not imply that $\tilde{f} \in L^\infty$ (see, e.g. [4]), the Hilbert transform still has very strong boundedness properties as can be seen in the following theorem, due to Zygmund [19].

Theorem A **(Zygmund)** *For $f \in L^\infty$ with $\|f\|_\infty \le \pi/2$ and $\lambda < 1$, there is a constant C_λ such that*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\lambda|\tilde{f}|} < C_\lambda, \tag{1.2}$$

and if f is continuous on $\mathbb{T}$, then

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\lambda|\tilde{f}|} < \infty \tag{1.3}$$

for all $\lambda < \infty$.

For the proof, see Corollary III.2.6 of [4]. It follows that

$$f = f_1 + f_2,\ \|f_1\|_\infty < \pi/2,\ f_2 \in C(\mathbb{T}) \implies \exp(\tilde{f}) \in L^1, \tag{1.4}$$

where $C(\mathbb{T})$ stands for the space of continuous functions on $\mathbb{T}$.

Let E be a measurable subset of $\mathbb{T}$ and define

$$\rho_E(z) = 2\chi_E(z) - 1 = \begin{cases} 1, & z \in E \\ -1, & z \notin \mathbb{T} \setminus E. \end{cases} \tag{1.5}$$

The Lebesgue measure of E is denoted by $|E|$. We note that the condition $\|f_1\|_\infty < \pi/2$ above is optimal as seen by considering an interval $E = [a, b] \subset (0, 2\pi)$ and showing that

$$\exp\left(\frac{\pi}{2}\tilde{\rho}_E(t)\right) = \exp\left(-\log\left|\sin\frac{t-b}{2}\right| + \log\left|\sin\frac{t-a}{2}\right|\right) = \left|\frac{\sin\frac{t-a}{2}}{\sin\frac{t-b}{2}}\right|, \quad (1.6)$$

which is not integrable in any neighborhood of b. More generally, it follows from (1.6) and Theorem III.2.7 of [4] that $\exp(\frac{\pi}{2}\tilde{\rho}_E) \notin L^1$ whenever E is a measurable subset of $\mathbb{T}$ with $0 < |E| < 2\pi$.

Notice that the conditions of (1.4) imply that if the function f is real valued and has jumps, then the size of each jump is strictly less than π, while the size of each jump of $\frac{\pi}{2}\rho_E$ is exactly π and $\exp(\frac{\pi}{2}\tilde{\rho}_E) \notin L^1$ with E as above. Motivated by the study of the Fredholm properties of Toeplitz operators, Shargorodsky [11] proved that if g is real valued and $\inf \mathcal{R}(g) > \pi/2$, where $\mathcal{R}(g)$ stands for the essential range of g, then $\exp(\widetilde{g\rho_E})$ is not integrable. These observations lead to the question of whether

$$f \in L^\infty \text{ and } f \geq \pi/2 \text{ a.e. } \implies \exp(\widetilde{f\rho_E}) \notin L^1. \quad (1.7)$$

Our main result answers this in the affirmative. Indeed, we give an elementary proof of the following result in Sect. 3.

Theorem 1 *Suppose that $f \in L^\infty$ with $f \geq \pi/2$ almost everywhere and $0 < |E| < 2\pi$. Then there is a positive constant C such that*

$$|E_\lambda| = \left|\{t : \widetilde{f\rho_E}(t) > \lambda\}\right| \geq Ce^{-\lambda} \quad (1.8)$$

for all $\lambda \geq 0$. In particular,

$$\exp\left(\widetilde{f\rho_E}\right) \notin L^1. \quad (1.9)$$

Remark 1 In the preceding theorem, the conditions on E and that $f \geq \pi/2$ almost everywhere are optimal—see Remark 3.

Remark 2 Notice that [11] contains the following local non-integrability result. Suppose that $0 < |E| < 2\pi$ and $\gamma \subset \mathbb{T}$ is an arc intersecting both E and $\mathbb{T} \setminus E$ in sets of positive measure and at least one of these sets is not an arc (modulo sets of measure zero). If $g \in L^\infty$ is real valued and $\inf \mathcal{R}(g) > \pi/2$, then $\exp(\widetilde{g\rho_E})$ is not integrable on γ.

Notice that Theorem 1 generalizes the preceding result for $\gamma = \mathbb{T}$. However, it remains an open problem whether our conclusion can be strengthened to that of [11] for the other arcs $\gamma \neq \mathbb{T}$ if we only assume that $g \geq \pi/2$ almost everywhere.

Previously in [9,15], sufficient conditions for exponential integrability of $\tilde{f}$ were obtained in terms of the modulus of continuity of f in L^p. In addition to these results and other intrinsic interest [4,7,18], the integrability of the exponential of conjugate functions plays an important role in the spectral theory of Toeplitz and related operators [10–12], scalar Riemann–Hilbert problems [13,14,17] and their applications.

2 Preliminaries

As indicated in the introduction, our approach is elementary and based on classical results of complex analysis which are briefly discussed in this section.

2.1 Poisson Integrals

For $f \in L^1$, denote by $P[f]$ the Poisson integral of f, that is,

$$P[f](z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta) f(\theta) d\theta \qquad (z \in \mathbb{D}), \tag{2.1}$$

where the Poisson kernel P_z is defined by

$$P_z(\theta) = \operatorname{Re} \frac{e^{i\theta}+z}{e^{i\theta}-z}.$$

Recall that $P[f]$ is harmonic in $\mathbb{D}$ and if the function f is continuous at $e^{i\theta}$, then

$$\lim_{z\to e^{i\theta}} P[f](z) = f(\theta) \tag{2.2}$$

(see Theorem I.1.3 of [5]). To deal with discontinuities at $e^{i\theta}$, define a cone Γ_α by

$$\Gamma_\alpha(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}| < \alpha(1-|z|)\}$$

for each $\alpha < 1$, and recall that a function $\varphi : \mathbb{D} \to \mathbb{C}$ is said to have nontangential limit $\varphi^*(e^{i\theta})$ at $e^{i\theta}$ if

$$\lim_{\Gamma_\alpha(e^{i\theta})\ni z\to e^{i\theta}} \varphi(z) = \varphi^*(e^{i\theta}) \tag{2.3}$$

for every $\alpha < 1$. Now, for any $f \in L^1$, if $u = P[f]$, then $u^* = f$ almost everywhere by Fatou's theorem.

Define

$$X_f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} f(\theta) d\theta \qquad (z \in \mathbb{D}).$$

Then

$$u(z) + i\tilde{u}(z) = X_f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} f(\theta)\, d\theta \tag{2.4}$$

for $z \in \mathbb{D}$, where $u = P[f]$ and $\tilde{u}$ is the harmonic conjugate function of u normalized so that $\tilde{u}(0) = 0$. By Fatou's theorem and Lemma III.1.1 of [4],

$$X_f^*(\theta) = f(\theta) + i\tilde{f}(\theta) \tag{2.5}$$

for almost every θ. For the integral representation of $\tilde{f}$ given in (1.1), see Lemma III.1.2 of [4]. In particular, it follows that the principal value in (1.1) exists almost everywhere. If $f \in L^\infty$, then

$$|\mathrm{Re}\, X_f(z)| = |u(z)| < \|f\|_\infty. \tag{2.6}$$

2.2 Harmonic and Subharmonic Functions

In one of the key steps of the proof of the main theorem, we consider the Dirichlet problem of finding a unique bounded harmonic function on a simply connected domain Ω with prescribed boundary values.

If g is a continuous real-valued function on $\partial\Omega$, the Dirichlet problem of finding the bounded harmonic function $u : \Omega \to \mathbb{R}$ such that $u = g$ on $\partial\Omega$ can be solved using the Poisson integral and the Riemann mapping theorem, which reduces the problem to the well-known case of the unit disk (see, e.g., [5]).

However, in our case, since the boundary functions are discontinuous (see Lemma 1 below), the following more general result is needed.

Theorem B *Let Ω be a simply connected domain and let g be a piecewise continuous function on $\partial\Omega$ with a finite number of discontinuities of the first kind at $\xi_1, \dots \xi_k$. Then there is at most one bounded harmonic function h on Ω such that $h = g$ on $\partial\Omega \setminus \{\xi_1, \dots, \xi_k\}$.*

If such a bounded harmonic function h exists, then

$$\inf_{\zeta \in \partial\Omega \setminus \{\xi_1,\dots,\xi_k\}} h(\zeta) \le h(z) \le \sup_{\zeta \in \partial\Omega \setminus \{\xi_1,\dots,\xi_k\}} h(\zeta)$$

for all $z \in \Omega$.

For the proof of the preceding result, see Theorems 5 and 6 of Sect. 42 of [8].

Regarding the values of a harmonic function in a domain Ω (a nonempty open connected set), we recall the maximum principle (Theorem 1.8 of [2]):

Theorem C *Suppose Ω is a domain, u is real-valued and harmonic on Ω, and u has a maximum or a minimum in Ω. Then u is constant.*

Lemmas 1 and 2 below are utilized in key steps of the proof of the main result.

Lemma 1 *Suppose that $f \in L^\infty$ with $f \ge \frac{\pi}{2}$ almost everywhere. Let $\lambda \ge 0$ and*

$$G_\lambda = \{|\mathrm{Re}\, z| < \pi/2\} \cup \{z : |\mathrm{Re}\, z| < \|f\|_\infty \text{ and } \mathrm{Im}\, z > \lambda + 1\}$$

be the domain in Fig. 1 and put $L = \partial G_\lambda \cap \{\operatorname{Im} z > \lambda + 1\}$. *Then there exists a unique bounded harmonic function* v_λ *on* G_λ *with the boundary values*

$$v_\lambda(z) = \begin{cases} 1, & z \in L, \\ 0, & z \in \partial G_\lambda \setminus (L \cup \{\pm\|f\|_\infty + i(\lambda+1)\}) \,. \end{cases} \tag{2.7}$$

Proof Let τ be a conformal map of G_λ onto the unit disk $\mathbb{D}$. Then each straight-line piece of the boundary ∂G_λ is mapped to an arc on the circle $\mathbb{T}$ (see Theorem II.3.4' of [6]). Now consider the Poisson integral of the function that equals 1 on the arcs to which τ maps L and 0 on the complementary arcs. The composition of the Poisson integral with τ gives the desired harmonic function. Uniqueness and the bounds

$$0 < v_\lambda(z) < 1 \quad \text{for } z \in G_\lambda \tag{2.8}$$

follow from Theorems B and C. □

The following characterization of subharmonic functions is also needed:

Theorem D *Let u be a function on* $\mathbb{D}$ *and suppose it satisfies the following conditions:*

(i) $\infty \le u < \infty$, $u \not\equiv -\infty$,
(ii) u is upper semi-continuous in $\mathbb{D}$,
(iii) for each $z_0 \in \mathbb{D}$, *there is an* r_0 *such that* $D(z_0, r_0) \subset \mathbb{D}$ *and*

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \tag{2.9}$$

for all $0 \le r \le r_0$.

Then u is subharmonic in $\mathbb{D}$.

For the proof of the preceding result, see Theorem II.13 of [16].

Lemma 2 *Let* λ, f, G_λ, *and* v_λ *be as in Lemma 1 and* $E \subset \mathbb{T}$ *be a measurable set with* $0 < |E| < 2\pi$. *Define*

$$X(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \big(f \rho_E\big)(\theta)\, d\theta \qquad (z \in \mathbb{D}),$$

where ρ_E *is defined in (1.5). Let* $\Omega_\lambda = X^{-1}(G_\lambda)$ *and define* $H_\lambda : \mathbb{D} \to \mathbb{C}$ *by*

$$H_\lambda(z) = \begin{cases} v_\lambda \circ X, & z \in \Omega_\lambda \\ 0, & z \in \mathbb{D} \setminus \Omega_\lambda. \end{cases}$$

Then H_λ *is continuous and subharmonic with* $0 \le H_\lambda < 1$.

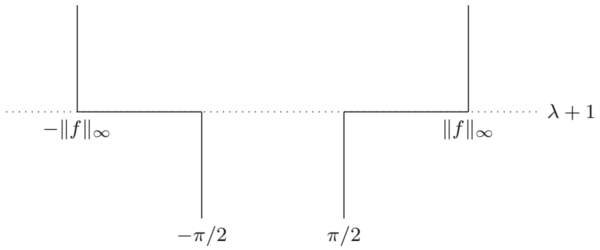

Fig. 1 The open set G_λ

Proof First, if $z \in \partial\Omega_\lambda \cap \mathbb{D}$, then $X(z) \notin G_\lambda$ but $X(z) \in \overline{G_\lambda \cap X(\mathbb{D})}$, and hence $X(z) \in \partial G_\lambda$. Further, since $|\mathrm{Re}\, X(z)| < \|f\|_\infty$ and $z \in \mathbb{D}$,

$$X(z) \in \{x + iy : |x| < \|f\|_\infty,\ y \le \lambda + 1\}$$

and so $v_\lambda(X(z)) = 0$ according to (2.7)). If $z_k \to z$ in Ω_λ, then, using the continuity of v_λ up to the point $X(z)$, $v_\lambda(X(z_k)) \to 0$, which implies that H_λ is continuous on $\mathbb{D}$. That $0 \le H_\lambda < 1$ follows from $0 < v_\lambda < 1$ (see (2.8)) and the definition of the function H_λ.

The local mean value inequality in (2.9) holds for each point of Ω_λ because $H_\lambda = v_\lambda \circ X$ is harmonic in Ω_λ, and it holds for each point of $\mathbb{D} \setminus \Omega_\lambda$ because $H_\lambda \ge 0$ equals zero there. Thus, by Theorem D, the function H_λ is subharmonic in $\mathbb{D}$. □

3 Proof of Theorem 1

Suppose that $f \in L^\infty$ with $f \ge \pi/2$ almost everywhere and $0 < |E| < 2\pi$. Let

$$X(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\big(f\rho_E\big)(\theta)\, d\theta \qquad (z \in \mathbb{D}),$$

where ρ_E is defined in (1.5). Then

$$X^* = f\rho_E + i\widetilde{f\rho_E}$$

almost everywhere according to (2.5).

The proof of (1.8) consists of several steps.

Step 1 Since $\mathrm{Re}\, X^* = f\rho_E \ge \pi/2$ almost everywhere on E and $\mathrm{Re}\, X^* \le -\pi/2$ almost everywhere on $\mathbb{T} \setminus E$, there is a point $w \in \mathbb{D}$ such that $|\mathrm{Re}\, X(w)| < \pi/2$.

Step 2 Let the open set G_λ (see Fig. 1) and $L = \partial G_\lambda \cap \{\mathrm{Im}\, z > \lambda + 1\}$ be defined as in Lemma 1.

Then Lemma 1 asserts that there exists a unique bounded harmonic function v_λ on G_λ with the boundary values

$$v_\lambda(z) = \begin{cases} 1, & z \in L, \\ 0, & z \in \partial G_\lambda \setminus (L \cup \{\pm\|f\|_\infty + i(\lambda+1)\}) . \end{cases}$$

Step 3 Let $\Omega_\lambda = X^{-1}(G_\lambda)$. Then, according to Lemma 2, the function $H_\lambda : \mathbb{D} \to \mathbb{C}$ defined by

$$H_\lambda(z) = \begin{cases} v_\lambda \circ X, & z \in \Omega_\lambda \\ 0, & z \in \mathbb{D} \setminus \Omega_\lambda \end{cases}$$

is continuous and subharmonic with $0 \le H_\lambda < 1$.

Step 4 There is a $C > 0$, independent of λ, such that

$$|E_\lambda| \ge C(v_\lambda \circ X)(w), \tag{3.1}$$

where E_λ is defined by (1.8) and w is defined in Step 1. Since H_λ is bounded, it trivially has a harmonic majorant, and hence, by Theorem I.6.7 of [4],

$$H_\lambda(z) \le \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} P_z(\theta) H_\lambda(re^{i\theta})\, d\theta \tag{3.2}$$

for $z \in \mathbb{D}$. It follows from (2.5) that, for almost every $\theta \in [0, 2\pi] \setminus E_\lambda$,

$$|\mathrm{Re}\, X^*(\theta)| = |f(\theta)| \ge \pi/2 \quad \text{and} \quad \mathrm{Im}\, X^*(\theta) \le \lambda.$$

Therefore, using the definition of H_λ and the properties of v_λ,

$$\lim_{r\to 1} H_\lambda(re^{i\theta}) = v_\lambda\left(X^*(\theta)\right) = 0.$$

Now, by (3.2) and Lebesgue's dominated convergence theorem,

$$\begin{aligned} H_\lambda(w) &\le \frac{1}{2\pi}\int_{E_\lambda} P_w(\theta)\, d\theta + \frac{1}{2\pi}\int_{[0,2\pi]\setminus E_\lambda} P_w(\theta) \lim_{r\to 1} H_\lambda(re^{i\theta})\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_w(\theta)\chi_{E_\lambda}(\theta)\, d\theta \le C\int_0^{2\pi} \chi_{E_\lambda}(\theta)\, d\theta = C|E_\lambda|, \end{aligned}$$

where the constant is independent of λ.

Step 5 It is difficult to obtain the desired estimate $(v_\lambda \circ X)(w) \ge Ce^{-\lambda}$ directly. Instead we estimate v_λ from below by another harmonic function g_λ, defined on a vertical strip, which allows us to compute g_λ explicitly in the next step.

Let g_λ be the bounded harmonic function in the strip $S = \{|\mathrm{Re}\, z| < \pi/2\}$ with the boundary values

$$g_\lambda(\pm\pi/2 + iy) = \begin{cases} 1, & y > \lambda + 2 \\ 0, & y < \lambda + 2. \end{cases}$$

Notice that $\lim_{y\to\infty} v_0(\pm\pi/2 + iy) = 1$ (which can be verified using a conformal map of Ω_0 onto $\mathbb{D}$). Therefore, since v_0 is continuous, Theorem C implies that there is a positive constant C such that

$$v_0(\pm\pi/2 + iy) \geq C \geq Cg_0(\pm\pi/2 + iy)$$

for all $y > 2$. Clearly $v_0(\pm\pi/2+iy) \geq 0 = Cg_0(\pm\pi/2+iy)$ for all $y < 2$. Thus, by Theorem B, $v_0(z) - Cg_0(z) \geq 0$ for all $z \in S$, and so for $\lambda > 0$ and $z \in S$,

$$v_\lambda(z) = v_0(z - i\lambda) \geq Cg_0(z - i\lambda) = Cg_\lambda(z).$$

Consequently, by (3.1),

$$|E_\lambda| \geq C(g_\lambda \circ X)(w),$$

where the constant C is independent of λ.

Step 6 Let $\tau = x + iy = X(w)$, so $|x| < \pi/2$ and $y \in \mathbb{R}$. We show that there is a constant $C > 0$, independent of λ, such that $g_\lambda(\tau) \geq Ce^{-\lambda}$, which completes the proof.

$$F(z) = \tan \tfrac{1}{2}z = -i\left(\frac{e^{\frac{1}{2}iz} - e^{-\frac{1}{2}iz}}{e^{\frac{1}{2}iz} + e^{-\frac{1}{2}iz}}\right)$$

defines a conformal mapping from S onto $\mathbb{D}$ with $F(ib) = i\tanh(\frac{1}{2}b) \to \pm i$, as $b \to \pm\infty$. Notice that

$$F^{-1}(z) = 2\arctan z = i\log\frac{1 - iz}{1 + iz}$$

for $z \in \mathbb{D}$. Thus, using the mapping $z \mapsto -iz$, we see that

$$z \mapsto i\log\frac{1 - \tan\frac{z}{2}}{1 + \tan\frac{z}{2}} = -i\log\frac{1 + \tan\frac{z}{2}}{1 - \tan\frac{z}{2}}$$

maps S conformally onto itself with $\pm\frac{\pi}{2} \mapsto (0, \mp\infty)$ and $(0, \infty) \mapsto \frac{\pi}{2}$. Therefore,

$$\begin{aligned} g_\lambda(z) &= \frac{1}{2} + \frac{1}{\pi}\mathrm{Re}\left(-i\log\frac{1 + \tan\frac{z - i(\lambda+2)}{2}}{1 - \tan\frac{z - i(\lambda+2)}{2}}\right) \\ &= \frac{1}{2} + \frac{1}{\pi}\arg\frac{1 + \tan\frac{z - i(\lambda+2)}{2}}{1 - \tan\frac{z - i(\lambda+2)}{2}}. \end{aligned}$$

To evaluate g_λ at $\tau = x + iy$, write $\tau_\lambda = \frac{\tau - i(\lambda+2)}{2}$ and notice that

$$g_\lambda(\tau) = \frac{1}{2} + \frac{1}{\pi} \arg \frac{1+\tan\tau_\lambda}{1-\tan\tau_\lambda} = \frac{1}{2} + \frac{1}{\pi} \arg\tan(\tau_\lambda + \pi/4)$$
$$= \frac{1}{2} + \frac{1}{\pi} \arctan \frac{\sinh(y-(\lambda+2))}{\sin(x+\pi/2)} = \frac{1}{2} - \frac{1}{\pi} \arctan \frac{\sinh(\lambda+2-y)}{\cos x}$$

using the formula $\arg\tan(a+ib) = \arctan\frac{\sinh 2b}{\sin 2a}$ and the fact that arctan is odd. Observe also that the expression for $g_\lambda(\tau)$ is valid in S, but not necessarily on ∂S and the first expression for $g_\lambda(z)$ is valid in $\bar{S}$. We first established the expression for $g_\lambda(z)$ with the desired boundary behavior on ∂S and then considered only the behavior inside S, where the expressions for $g_\lambda(z)$ and $g_\lambda(\tau)$ coincide. Since $\arctan a = \frac{\pi}{2} - \arctan a^{-1}$ for $a > 0$, we have, for $\lambda + 2 > y$,

$$g_\lambda(\tau) = \frac{1}{\pi} \arctan \frac{\cos x}{\sinh(\lambda+2-y)},$$

which gives the estimate. This completes the proof of (1.8).

It remains to prove (1.9). As in Sect. 4 of Chapter I in [4], consider the distribution function

$$m(\lambda) = \left|\left\{t : \exp\left(\widetilde{f\rho_E}(t)\right) > \lambda\right\}\right|$$

for $\lambda > 0$. Notice that for $\lambda \geq 1$,

$$m(\lambda) = \left|E_{\log\lambda}\right|$$

(see (1.8) for the definition of $E_{\log\lambda}$), and so by Lemma I.4.1 in [4],

$$\int \exp(\widetilde{f\rho_E}(\theta))\, d\theta = \int_0^\infty m(\lambda)\, d\lambda \geq \int_1^\infty \left|E_{\log\lambda}\right| d\lambda. \tag{3.3}$$

It remains to combine (1.8) with (3.3).

4 Further Remarks

In this section we provide remarks and examples related to Theorem 1. We show first that the conditions in the theorem are optimal.

Remark 3 In Theorem 1, (i) the condition on E is optimal, and (ii) the condition that $f \geq \pi/2$ almost everywhere is optimal.

Proof (i) Let $E = [0, 2\pi]$ and $f = \pi/2$ on E. Then $f\rho_E = f$ and so trivially $e^{\widetilde{f\rho_E}} \in L^1$ by (1.3).

Further, by Corollary III.1.8 of [7],

$$|E_\lambda| = \left|\{t : |\widetilde{f\rho_E}(e^{it})| > \lambda\}\right| = \left|\{t : |\tilde{\chi}_E(e^{it})| > 2\lambda/\pi\}\right| < 10\pi e^{-2\lambda}, \quad (4.1)$$

which implies that there is no constant C for which (1.8) holds in this case.

(ii) Suppose that $\pi/4 < f < \pi/2$ on an interval $I = (a, b) \subset (0, 2\pi)$ and $f = \pi/2$ on $[0, 2\pi] \setminus I$. Let $E = [0, a+\delta] \cup [b-\delta, 2\pi]$ for some small $\delta > 0$. Then $\exp(\widetilde{f\rho_E})$ is integrable by Zygmund's Theorem A because the gap in the essential range of $f\rho_E$ is strictly less than π. Consequently, the condition is optimal for (1.9), and it must also be optimal for (1.8) because it was used to prove (1.9). □

Remark 4 In addition to the example in the previous proof, there are functions f which are not constant and still satisfy $|E_\lambda| \lesssim e^{-\lambda}$ as in (4.1). Indeed, let $0 < |E| < 2\pi$ and let f be Hölder with $f(\theta) = \pi/2$ for all $\theta \in \partial E$. If $g = (f - \pi/2)\rho_E$, then it is not difficult to see that g is Hölder and hence $\tilde{g}$ is Hölder. Also,

$$\widetilde{f\rho_E} - \tilde{g} = \tfrac{\pi}{2}\tilde{\rho}_E.$$

Thus, by Exercise VI.18 of [4],

$$\left|\{t : |\widetilde{f\rho_E}(t) - \tilde{g}(t)| > \lambda\}\right| = \left|\{t : |\tfrac{\pi}{2}\tilde{\rho}_E(t)| > \lambda\}\right| \lesssim e^{-\lambda},$$

which implies that $|E_\lambda| = \left|\{t : |\widetilde{f\rho_E}(t)| > \lambda\}\right| \lesssim e^{-\lambda + \|\tilde{g}\|_\infty} \le Ce^{-\lambda}$ for some constant C.

In fact, (0.2) of [18], when $\psi = \widetilde{f\rho_E} = \tilde{g} + \frac{\pi}{2}\tilde{\rho}_E$, implies the stronger result

$$\sup_I \frac{|\{t \in I : |\psi(t) - \psi_I| > \lambda\}|}{|I|} \lesssim e^{-\lambda}. \quad (4.2)$$

Notice that the converse is not true, however, that is, (4.2) does not imply that

$$\psi = u + \tilde{v}, \quad u \in L^\infty, \|v\|_\infty \le \pi/2 \quad (4.3)$$

(see Wolff's counterexample on page 52 of [18]).

Open Problem 1 In the preceding remark, when $\psi = \frac{1}{2}\log|H|$ with H univalent and zero free, Baernstein [3,18] posed a question of whether (4.2) implies (4.3). This seems to be still open.

Our next example concerns outer functions from the theory of Hardy spaces. Recall that an outer function is a function G on the unit disk which can be written in the form

$$G(z) = \alpha \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log\varphi(t)\, dt\right), \quad \alpha \in \mathbb{C},\ z \in \mathbb{D}, \quad (4.4)$$

where $|\alpha| = 1$ and φ is a positive measurable function on $\mathbb{T}$ such that $\log\varphi \in L^1$. Similarly to (2.4),

$$G(z) = e^{u(z)+iv(z)}, \quad z \in \mathbb{D}, \tag{4.5}$$

where u is the Poisson integral of $\log \varphi$ and v is the harmonic conjugate function of u so that $e^{iv(0)} = \alpha$.

Let $0 < p < \infty$ and let f be analytic in $\mathbb{D}$. Then the function f is in the Hardy space H^p if

$$\sup_{r<1} \int_0^{2\pi} |f(re^{it})|^p dt < \infty.$$

Notice that $G \in H^p$ if and only if $\varphi \in L^p$ (see, e.g., Sect. II.4 of [4]). We can now use Theorem 1 to determine when certain outer functions are not in H^p as shown in the following example.

Example 1 Given a real-valued function f in L^∞, define

$$\Phi_f(z) = \exp\left(\frac{i}{4\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} f(t)\, dt\right), \quad |z| < 1.$$

Using (4.5), it is easy to see that Φ_f and its inverse Φ_f^{-1} are both outer functions. Denote by $\mathcal{R}(f)$ the essential range of f as before. Let

$$I \subset [\operatorname{ess\,inf}_{t\in\mathbb{T}} f(t), \operatorname{ess\,sup}_{t\in\mathbb{T}} f(t)]$$

be an interval such that $I \cap \mathcal{R}(f) = \emptyset$ and $|I| \geq \frac{2\pi}{p}$. Then $\Phi_f, \Phi_f^{-1} \notin H^p$. In particular, $\Phi_f^{\pm 1} \notin H^2$ if $\mathcal{R}(f)$ has a gap of length $\geq \pi$.

To see this, notice first that (similarly to (2.5))

$$(\Phi_f^{\pm 1})^*(t) = \exp\left(\mp\frac{1}{2}\left(\tilde{f}(t) - if(t)\right)\right),$$

so $|(\Phi_f^{\pm 1})^*(t)| = e^{\mp\frac{1}{2}\tilde{f}(t)}$. Since Φ_f is an outer function, it follows from Theorem 1 that $\Phi_f, \Phi_f^{-1} \notin H^p$ if $\frac{p}{2}|I| \geq \pi$.

To illustrate the effect of jumps in relation to exponential integrability (see Example 2), the following lemma will be needed. Its proof is included for completeness because we have not found it in the literature.

Lemma 3 *For* $0 < x < 2\pi$, *let*

$$g(x) = \sum_{n=2}^{\infty} \frac{\cos nx}{n \log n}.$$

Then

$$g(x) = \log\log\frac{1}{x} + B + O\left((\log\log\frac{1}{x})^{1/3}(\log\frac{1}{x})^{-2/3}\right) \tag{4.6}$$

as $x \to 0+$, *where* B *is a positive constant.*

Proof Define $B_x(y) = \sum_{2\le n\le y} \cos nx$. Then, for $0 < x < \pi$ and $y > 2$, $|B_x(y)| \le \frac{\pi}{x} + 1$. Therefore, integrating by parts,

$$\sum_{n=N}^{M} \frac{\cos nx}{n\log n} = \int_N^M \frac{dB_x(y)}{y\log y} = \frac{B_x(M)}{M\log M} - \frac{B_x(N)}{N\log N} + \int_N^M \frac{B_x(y)(\log y+1)}{y^2|\log y|^2}dy,$$

and so

$$\left|\sum_{n=N}^{\infty} \frac{\cos nx}{n\log n}\right| \le \frac{\frac{\pi}{x}+1}{N\log N} + \left(\tfrac{\pi}{x}+1\right)\int_N^\infty \frac{\log y+1}{y^2\log^2 y}dy$$
$$\le 2\frac{\frac{\pi}{x}+1}{N\log N} \le \frac{4\pi}{xN\log N}.$$

Suppose now that $x < \frac{\pi}{2N}$. Then

$$\sum_{n=2}^{N} \frac{1}{n\log n} \ge \sum_{n=2}^{N} \frac{\cos nx}{n\log n} \ge \cos Nx \sum_{n=2}^{N} \frac{1}{n\log n} \ge \left(1 - \tfrac{x^2N^2}{2}\right)\sum_{n=2}^{N} \frac{1}{n\log n}.$$

Since

$$\sum_{k=2}^{n} \frac{1}{k\log k} = \log\log n + B + O\left(\frac{1}{n\log n}\right),$$

where B is a constant (see Exercise 8.20 of [1]), we get for $x < \frac{\pi}{2N}$,

$$g(x) = \log\log N + B + O\left(\frac{1}{N\log N}\right) + O\left(x^2N^2\log\log N\right) + O\left(\frac{1}{xN\log N}\right)$$

as $N \to \infty$.

Choosing $N = \frac{1}{x}(\log\frac{1}{x}\log\log\frac{1}{x})^{-1/3}$, we obtain (4.6) as $x \to 0+$. □

Remark 5 In the following example, for small values of $|x|$, we only need the following consequence of the preceding lemma:

$$\sum_{n=2}^{\infty} \frac{\cos nx}{n\log n} \ge C\log\log(1/|x|),$$

which can also be obtained using Theorem V.1.5 of [19].

Example 2 Let $t_0 \in (0, 2\pi)$. For $0 < \delta < 1$ define

$$g(t) = \begin{cases} \pi/2 & t_0 - \delta < t < t_0 \\ -\pi/2 & t_0 < t < t_0 + \delta \end{cases}$$

and suppose that g is Hölder continuous elsewhere. By Theorem V.1.3 of [20], the series

$$h(t) = 2\sum_{n=2}^{\infty} \frac{\sin n(t-t_0)}{n\log n}$$

converges uniformly and defines a continuous function on $[0, 2\pi]$. It is well known that

$$\tilde{h}(t) = -2\sum_{n=2}^{\infty} \frac{\cos n(t-t_0)}{n\log n},$$

which is continuous on $[0, 2\pi] \setminus \{t_0\}$ (see Theorem I.2.6 of [20]). Define $f = g + h$. Then $\lim_{t\to t_0\pm} f(t) = \mp\pi/2$, so f is piecewise continuous with only one jump, which is of size π. We want to determine whether $\exp(\tilde{f})$ is integrable. Obviously we cannot use Zygmund's Theorem A. Notice also that we cannot apply Theorem 1 because we do not know without further inspection whether $f \geq \pi/2$ a.e. on $(0, t_0)$ and $f \leq -\pi/2$ a.e. on $(t_0, 2\pi)$.

Now, similarly to (1.6), and using the fact that g is Hölder on $[0, 2\pi] \setminus \{t_0\}$, there is a constant $C > 0$ such that

$$\tilde{g}(t) \leq -\log\left|\sin\frac{t-t_0}{2}\right| + C$$

for $0 \leq t \leq 2\pi$. To estimate $\tilde{h}$ near t_0, notice first that Lemma 3 implies that

$$\tilde{h}(t) \leq -2\log(\log|t-t_0|^{-1})$$

for t sufficiently close to t_0. Therefore, for some constants C, we have

$$\begin{aligned} e^{\tilde{f}(t)} = e^{\tilde{g}(t)}e^{\tilde{h}(t)} &\leq C\exp\left(-\log|\sin\frac{t-t_0}{2}|\right)\exp\left(-2\log\log\frac{1}{|t-t_0|}\right) \\ &= C\left|\sin\frac{t-t_0}{2}\right|^{-1}\left(\log\frac{1}{|t-t_0|}\right)^{-2} \leq C|t-t_0|^{-1}(\log|t-t_0|)^{-2}, \end{aligned}$$

which is integrable in a neighborhood of t_0, and hence $e^{\tilde{f}} \in L^1$.

Now, of course, by Theorem 1, the integrability of $e^{\tilde{f}}$ means that f has values in $(-\pi/2, \pi/2)$ on a set of positive measure. In fact, by V.2.13 of [19], the function h is positive on $(t_0, t_0+\epsilon)$ for some $\epsilon > 0$ and hence negative on $(t_0-\epsilon, t_0)$ (as an odd function), and so indeed $f < \pi/2$ on $(t_0-\epsilon, t_0)$ and $f > -\pi/2$ on $(t_0, t_0+\epsilon)$.

Remark 6 We can use the previous example to construct a function $f \in L^\infty$ such that $\|f\|_\infty = \pi/2$, $|f| < \pi/2$ and $e^{\tilde{f}} \in L^1$. This should be compared with Zygmund's result in (1.2).

Open Problem 2 In Theorem 1, it is assumed that $f \in L^\infty$ as in Zygmund's Theorem A. It is natural to ask whether the conclusion of Theorem 1 remains true for unbounded functions.

We finish this section with a connection between the estimate in (1.8) and the distance in BMO to L^∞ (see Sect. VI.6 of [4]).

Suppose that the conditions of Theorem 1 are satisfied, that is, $f \in L^\infty$ with $f \geq \pi/2$ almost everywhere and $0 < |E| < 2\pi$. If $\|g\|_\infty < \pi/2$, then by Theorems A and 1, $\widetilde{f\rho_E} - \tilde{g} \notin L^\infty$, and so $\mathrm{dist}(\widetilde{f\rho_E}, L^\infty) \geq \pi/2$, where

$$\mathrm{dist}(\varphi, L^\infty) = \inf\{\|g\|_\infty : \varphi - \tilde{g} \in L^\infty\},$$

which is equivalent to $\inf_{g\in L^\infty} \|\varphi - g\|_{\mathrm{BMO}}$ (see, e.g., page 250 in [4]). By Corollary VI.6.6, there is no $\epsilon \in (0, 1)$ such that

$$\sup_I \frac{|\{t \in I : |\widetilde{f\rho_E}(t) - (\widetilde{f\rho_E})_I| > \lambda\}|}{|I|} \leq e^{-\lambda/\epsilon} \tag{4.7}$$

for all $\lambda \geq 0$, where the supremum is taken over all arcs $I \subset \mathbb{T}$ and the average φ_I is defined by $\varphi_I = \frac{1}{|I|}\int_I \varphi$ for $\varphi \in L^1$.

The same conclusion also follows directly from (1.8) if $f \geq \pi/2$, which is no surprise because the requirement that f has a gap in its essential range is a stronger assumption than $\mathrm{dist}(\widetilde{f\rho_E}, L^\infty) \geq \pi/2$. Indeed, assume that (1.8) holds and (4.7) does not hold, so that there exists $\epsilon \in (0, 1)$ satisfying the estimate (4.7). Denote $h = f\rho_E$ so that $\tilde{h}_0 = \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}$, and choose $s \geq 0$ so large that $\lambda = \tilde{h}_0 + s \geq 0$. Then there exists a constant $C > 0$ so that

$$\begin{aligned} Ce^{-\lambda} &\leq \frac{1}{2\pi}|\{\theta \in [0, 2\pi] : \tilde{h}(\theta) > \lambda\}| = \frac{1}{2\pi}|\{\theta \in [0, 2\pi] : \tilde{h}(\theta) - \tilde{h}_0 > s\}| \\ &\leq \sup_{I\subset\mathbb{T}} \frac{1}{|I|}|\{\theta \in I : |(\tilde{h} - \tilde{h}_I)(\theta)| > s\}| \leq e^{-\frac{s}{\epsilon}}, \end{aligned}$$

so we have $Ce^{-\lambda} = Ce^{-(\tilde{h}_0+s)} \leq e^{-\frac{s}{\epsilon}}$. Therefore $Ce^{-\tilde{h}_0} \leq e^{(1-\frac{1}{\epsilon})s} \to 0$ as $s \to \infty$, which is a contradiction.

5 Complex-Valued Functions

While real-valued functions are of particular importance in the study of exponential integrability of their conjugate functions, especially in connection with applications, such as Riemann–Hilbert problems and spectral theory of Toeplitz operators above, it would also be of interest to consider the case of complex-valued functions. Indeed, as in Zygmund's Theorem A, we may consider a complex-valued $f \in L^\infty$ and ask under what conditions is $\exp(|\widetilde{f\rho_E}|)$ not integrable. As in the proof of Theorem 1, we can define $m(\lambda) = |\{t : \exp(|\widetilde{f\rho_E}(t)|) > \lambda\}|$ and show that if there is a constant $|\{t : |\widetilde{f\rho_E}(t)| > \lambda\}| \geq Ce^{-\lambda}$ for all $\lambda \geq 0$, then $\exp(|\widetilde{f\rho_E}|)$ is not integrable.

However, in the complex case, the function $f\rho_E$ no longer has a similar (geometric) meaning as in the real case where it can be related to a gap in the essential range. For this reason, we say that a set $A \subset \mathbb{C}$ has a gap of size $g > 0$ if $A = B \cup C$ for some sets B and C of positive measure with $\mathrm{dist}(A, B) \geq g$. With this, we can state (1.9) in Theorem 1 as follows: If $f \in L^\infty$ is real and $\mathcal{R}(f)$ has a gap of size at least π, then $\exp(\tilde{f})$ is not integrable.

Open Problem 3 Given a complex-valued function f in L^∞, find a converse to Zygmund's Theorem A.

It may be useful to try to relate the exponential integrability of $\tilde{f}$ to the size of the gap in the essential range of f as in the real case.

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