The Berezin transform of Toeplitz operators on the weighted Bergman space

Potential Analysis
https://doi.org/10.1007/s11118-021-09915-2

The Berezin Transform of Toeplitz Operators on the Weighted Bergman Space

Cezhong Tong[1] **· Junfeng Li**[1] **· Hicham Arroussi**[2]

Received: 23 September 2019 / Accepted: 14 February 2021 / Published online: 08 April 2021

Abstract
In this paper, we obtain some interesting reproducing kernel estimates and some Carleson properties that play an important role. We characterize the bounded and compact Toeplitz operators on the weighted Bergman spaces with Békollé-Bonami weights in terms of Berezin transforms. Moreover, we estimate the essential norm of them assuming that they are bounded.

Keywords Békollé weights · Carleson measures · Toeplitz operators

Mathematics Subject Classification (2010) Primary: 42B35 · Secondary: 32A36

1 Introduction and Results

Let $\mathbb{C}$ be the complex plane and $D(0,r) := \{z \in \mathbb{C} : |z| < r\}$ for $r > 0$ the Euclidean open disc with center 0 and radius r. We denote by the unit disc $\mathbb{D} := D(0,1)$ for short. If μ is a positive measure on $\mathbb{D}$ and $p > 0$, we denote $L^p(\mu)$ the Lebesgue space over $\mathbb{D}$ with respect to μ. That is, $L^p(\mu)$ consists of all functions f defined on $\mathbb{D}$ for which

$$\|f\|_{L^p(\mu)} := \left[\int_{\mathbb{D}} |f(z)|^p \mathrm{d}\mu(z)\right]^{1/p} < \infty.$$

When $p \geq 1$, $\|\cdot\|_{L^p(\mu)}$ defines a norm and $L^p(\mu)$ becomes a Banach space.

Cezhong Tong was supported in part by China Scholarship Fund and National Natural Science Foundation of China (Grant No. 11301132).

✉ Hicham Arroussi
arroussihicham@yahoo.fr

Cezhong Tong
ctong@hebut.edu.cn; cezhongtong@hotmail.com

Junfeng Li
hebutlijunfeng@126.com

[1] Department of Mathematics, Hebei University of Technology, 300401, Tianjin P.R., China

[2] Department of Mathematics, University of Reading, RG6 6AX, Reading, England

Let $\mathrm{d}A$ denote the Lebesgue area measure on $\mathbb{D}$. If u is a positive locally integrable function on $\mathbb{D}$, i.e. positive $u \in L^1_{loc}(\mathrm{d}A)$, let $L^p(u)$ denote the space of measurable functions on $\mathbb{D}$ that are pth power integrable with respect to $u\mathrm{d}A$. That is

$$\|f\|_{L^p(u)} := \left(\int_{\mathbb{D}} |f(z)|^p u(z) \mathrm{d}A(z) \right)^{1/p} < \infty.$$

The Bergman space $A^p(u)$ is defined to be a subspace of analytic functions in $L^p(u)$ with $L^p(u)$-norm. We write $A^p = A^p(1)$ for short. The most common reproducing kernel for the unit disc has the form

$$K_w(z) = \frac{1}{(1 - \bar{w}z)^2},$$

for $w, z \in \mathbb{D}$, and it corresponds to the space A^2.

The following notations will be used throughout the paper. For a weight u and $E \subset \mathbb{D}$, we set $u(E) = \int_E u\mathrm{d}A$, $A(E) = \int_E \mathrm{d}A$. We denote by

$$\langle f \rangle_E^{\mathrm{d}\mu} := \frac{\int_E f(z)\mathrm{d}\mu(z)}{\mu(E)}$$

for integrable f and measure μ.

If we define P by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^2} \mathrm{d}A(w).$$

The problem of characterizing the weights for which the Bergman projection P is a bounded orthogonal projection from $L^p(u)$ to $A^p(u)$ was solved by Békollé and Bonami [1, 2]. They found that these weights are precisely $u \in B_p$.

B_p condition Let $S(a)$ be the set

$$S(a) = \left\{ \frac{a - z}{1 - \bar{a}z} : \mathrm{Re}\,(\bar{a}z) \leq 0 \right\}.$$

We say u satisfies B_p condition, or $u \in B_p$, if

$$[u]_{B_p} := \sup_{a \in \mathbb{D}} \langle u \rangle_{S(a)}^{\mathrm{d}A} \left(\left\langle u^{-p'/p} \right\rangle_{S(a)}^{\mathrm{d}A} \right)^{p-1} \lesssim 1$$

where $1/p + 1/p' = 1$. Recently, the sharp estimates for the L^p-continuity of the Bergman projection are investigated in [9] and [10] respectively.

The inner product of the Hilbert space $A^2(u)$ is given by

$$\langle f, g \rangle_{A^2(u)} = \int_{\mathbb{D}} f(w)\overline{g(w)} u(w) \mathrm{d}A(w),$$

where $f, g \in A^2(u)$. The reproducing kernel of $A^2(u)$ will be denoted by $K(z, w)$. It is well known that $K(z, w) = \overline{K(w, z)}$. If L_z is the point evaluation at $z \in \mathbb{D}$, that is $L_z f = f(z)$ for every $f \in A^2(u)$. It follows by the Riesz representation that

$$K(z, z) = \langle K(., z), K(., z) \rangle_{A^2(u)} = \|K(., z)\|^2_{A^2(u)} = \|L_z\|^2_{A^2(u) \to \mathbb{C}}.$$

Given a positive Borel measure μ on $\mathbb{D}$, the Toeplitz operator T_μ associated with μ on $A^2(u)$ is the linear transformation defined by

$$T_\mu f(z) := \int_{\mathbb{D}} f(w) K(z, w) \mathrm{d}\mu(w), \quad z \in \mathbb{D}.$$

Let μ be a finite positive Borel measure on $\mathbb{D}$ that satisfies the condition

$$\int_{\mathbb{D}} |K(\xi, z)|^2 \mathrm{d}\mu(\xi) < \infty.$$

Then the Toeplitz operator T_μ is well-defined on $A^2(u)$.

Recall that the pseudohyperbolic metric $d : \mathbb{D} \times \mathbb{D} \to [0, 1)$ is defined by

$$d(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|.$$

Denote by

$$\Delta(z, r) := \{w \in \mathbb{D} : d(z, w) < r\}$$

the pseudohyperbolic disk centered at z with radius r. For a finite positive Borel measure μ on $\mathbb{D}$ and $r > 0$, the average function $\widehat{\mu_r}$ is defined as

$$\widehat{\mu_r}(z) = \frac{\mu(\Delta(z, r))}{u(\Delta(z, r))}, \quad z \in \mathbb{D}.$$

It is well known that the Berezin trasform plays a role in the theory of Toeplitz operator. The Berezin transform of the Toeplitz operator T_μ is given by

$$\tilde{\mu}(z) := \langle T_\mu k_z, k_z \rangle_{A^2(u)}, \quad z \in \mathbb{D},$$

where $k_z(w) := K(w, z)/\|K(\cdot, z)\|_{A^2(u)}$ is the normalized reproducing kernel of $A^2(u)$. By a straightforward computation one has

$$\langle T_\mu f, g \rangle_{A^2(u)} = \langle f, g \rangle_{L^2(\mathrm{d}\mu)}. \tag{1.1}$$

It follows that the Berezin transform $\tilde{\mu}$ can be formulated by

$$\tilde{\mu}(z) = \int_{\mathbb{D}} |k_z(w)|^2 \mathrm{d}\mu(w), \quad z \in \mathbb{D}.$$

Constantin [5] characterized the Toeplitz operator on $A^2(u)$ in terms of the Carleson measure. The motivation of this paper is to characterize the Toeplitz operator in terms of its Berezin transform. Now we are in the position to state our main theorems.

Theorem 1.1 *Let $p_0 > 1$ and $u \in B_{p_0}$. Suppose that $\delta \in (0, 1)$ is the one in Theorem 2.7 and $0 < r \le \delta$. The following assertion are equivalent:*

(i) *The Toeplitz operator T_μ is bounded on $A^2(u)$.*
(ii) *$\tilde{\mu}$ is bounded on $\mathbb{D}$.*
(iii) *$\widehat{\mu_r}$ is bounded on $\mathbb{D}$.*

The following theorem charaterizes the compact Toeplitz operators on $A^2(u)$ with Békollé-Bonami weights

Theorem 1.2 *Let $p_0 > 1$ and $u \in B_{p_0}$. Suppose that $\delta \in (0, 1)$ is the one in Theorem 2.7 and $0 < r \le \delta$. The following assertions are equivalent:*

(i) *The Toeplitz operator T_μ is compact on $A^2(u)$.*
(ii) $\lim_{|z| \to 1} \tilde{\mu}(z) = 0$.
(iii) $\lim_{|z| \to 1} \widehat{\mu_r}(z) = 0$.

Next, we will study the Schatten class of Toeplitz operators $T_\mu \in \mathcal{S}_p\left(A^2(u)\right)$ in terms of the Berezin transform. Recall that if T is a compact operator on a Hilbert space H, then there are orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in H such that

$$Tx = \sum_{n=1}^{\infty} s_n \langle x, e_n\rangle_H \sigma_n, \quad x \in H,$$

where $s_n = s_n(T)$ is the nth singular value of T. The Schatten class $\mathcal{S}_p = \mathcal{S}_p(H)$ consists of those compact operators T on H for which the singular numbers sequence $\{s_n\}$ of T belongs to ℓ^p, that is $\sum_n |s_n|^p < \infty$.

Theorem 1.3 *Let $p > 1$, $p_0 > 1$ and $u \in B_{p_0}$. Suppose that μ is a positive Borel measure on $\mathbb{D}$ such that the Toeplitz operator T_μ is compact on $A^2(u)$. Then $T_\mu \in \mathcal{S}_p\left(A^2(u)\right)$ if and only if $\tilde{\mu} \in L^p(d\lambda_u)$ where $d\lambda_u = \|K_z\|^2 u(z) dA$.*

Let $\mathcal{K}$ be the set of all compact operators on a Banach space $\mathcal{B}$. For any bounded linear operator $T : \mathcal{B} \to \mathcal{B}$, the essential norm of T is defined by

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}\}.$$

It is clear that $\|T\|_e = 0$ if and only if $T \in \mathcal{K}$. Finally, we show the conditions for Toeplitz operators to be compact, see the above theorem, in term of the essential norm estimates because essential norm estimates give us a further information. The essential norm of a bounded operator is the distance from the operator to the space of the compact operators.

Theorem 1.4 *Let μ be a finite positive Borel measure on $\mathbb{D}$. Suppose that T_μ is a bounded operator on $A^2(u)$. Then, one has,*

$$\|T_\mu\|_e \simeq \limsup_{|z|\to 1} \tilde{\mu}(z) \simeq \limsup_{|z|\to 1} \widehat{\mu_r}(z).$$

Throughout the paper, we use the following notations:

- $Q_1 \lesssim Q_2$ means that there is a constant $C > 0$ (independent of the key variable(s)) such that $Q_1 \leq C Q_2$;
- $Q_1 \simeq Q_2$ if both $Q_1 \lesssim Q_2$ and $Q_2 \lesssim Q_1$.

2 Preliminaries and Basic Properties

The pseudohyperbolic metric obeys the following so-called *strong triangle inequality*:

$$\rho(z, w) \leq \frac{\rho(z, \zeta) + \rho(\zeta, w)}{1 + \rho(z, \zeta)\rho(\zeta, w)}$$

for all $z, w, \zeta \in \mathbb{D}$. Furthermore, if $0 < r < 1$, then whenever $z, w \in \mathbb{D}$ with $\rho(z, w) < r$,

$$1 - |z| \simeq 1 - |w| \simeq |1 - \bar{w}z| \tag{2.1}$$

and for all $\zeta \in \mathbb{D}$

$$\left|\frac{1 - \bar{\zeta}z}{1 - \bar{\zeta}w}\right| \simeq 1$$

where the constants involved depend only on r. We will denote by

$$\Delta(z, r) := \{w \in \mathbb{D} : \rho(z, w) < r\}$$

the pseudohyperbolic disk centered at z with radius r.

We will also use the following class of weights which is denoted by C_p. For $p>1$, a positive locally integrable weights u belongs to C_p, or say u satisfies C_p condition if

$$[u]_{C_p} := \sup_{z\in\mathbb{D}} \langle u\rangle^{\mathrm{d}A}_{\Delta(z,r)} \left(\left\langle u^{-p'/p}\right\rangle^{\mathrm{d}A}_{\Delta(z,r)}\right)^{p-1} \lesssim 1$$

where $1/p+1/p'=1$. Condition C_p seems to depend on a choice of $r<1$, but it is known that the same class of weights is obtained for any $r\in(0,1)$ and $B_p\subset C_p$. To see this, we note that for a given r, there is a $a'\in\mathbb{D}$ such that $\Delta(a,r)\subset S\left(a'\right)$ with comparable volumes, for more details see [7].

It is not hard to see that $S(a)$ is "equivalent" to the set $S(\zeta,h)$ for $\zeta=a/|a|\in\partial\mathbb{D}$ and $h=1-|a|$ in the sense that $S(\zeta,h)\subset S(a)\subset S(\zeta,2h)$, where

$$S(\zeta,h)=\{z\in\mathbb{D}:|1-z\bar{\zeta}|<h\}.$$

See more details in [7].

The point evaluations on $A^p(u)$ are bounded linear functionals for $p>0$. To be precise, we have the following estimate.

Lemma 2.1 (Lemma 3.1 in [7]) *If $p_0>1$, $p>0$, $0<r<1$ and a weight $u\in C_{p_0}$, we have*

$$|f(z)|^p \le Cu(\Delta(z,r))^{-1}\int_{\Delta(z,r)}|f(w)|^p u(w)dA(w),$$

where the constant $C>0$ depends on r, p and the C_{p_0} constant $[u]_{C_{p_0}}$.

In the Békollé setting, Bergman metric balls have comparable weighted areas when their centers are close.

Lemma 2.2 (Lemma 2.2 in [5]) *Suppose $u\in C_p$ for some $p>1$. Let $t,s\in(0,1)$, and $z,w\in\mathbb{D}$ with $\rho(z,w)<r$ for some $r>0$. Then we have*

$$u(\Delta(z,t))\simeq u(\Delta(w,s)),$$

where the constant is independent of z and w.

Similarly, if $u\in B_{p_0}$, it is worthy to be noted that

$$u(\Delta(a,r))\simeq u\left(S\left(a'\right)\right) \tag{2.2}$$

whenever $\Delta(a,r)\subset S\left(a'\right)$ with comparable volumes. To interested readers we can refer [7] and Lemma 5.23 in [13] for more details.

For $s>0$ and $0<r<1$, we denote by

$$G^s_w(z)=\frac{1}{(1-z\bar{w})^s}.$$

Test functions play a crucial role in our proofs. Constantin [4] gives an estimate of the the norm of G^s_w in terms of the weighted area of Euclidean disks inside $\mathbb{D}$. We can adopt an alternative method to estimate the norm of G^s_w in terms of the weighted area of $S(a)$ (or $S(\zeta,h)$ equivalently). Our method relies on a popular decomposition of $\mathbb{D}$ which is used repeatedly in many papers. See Theorem 1 in [12] for instance. The first two authors obtain the same estimate on the unit ball by an analogue method, see [11]. For the sake of clarity, we reprove it here.

Lemma 2.3 *Let $p > 0$, $p_0 > 1$ and the weight $u \in B_{p_0}$. We have*

$$\frac{u(S(w))^{\frac{1}{p}}}{(1-|w|)^s} \lesssim \|G_w^s\|_{L^p(u)} \lesssim \frac{u(S(w))^{\frac{1}{p}}}{(1-|w|)^{\max\{2p_0/p,s\}}} \tag{2.3}$$

where the constant involved is independent of $w \in \mathbb{D}$.

Proof If $z \in S(w) \subset S(w/|w|, 2(1-|w|))$ then

$$2(1-|w|) \geq \left|1 - z\frac{\bar{w}}{|w|}\right| \geq |1-z\bar{w}| - \left|z\left(\bar{w} - \frac{\bar{w}}{|w|}\right)\right| \geq |1-z\bar{w}| - (1-|w|).$$

Rearranging this inequality, we have $1-|w| \geq |1-z\bar{w}|/3$, and it follows immediately that

$$\frac{u(S(w))}{(1-|w|)^{ps}} \lesssim \int_{S(w)} \frac{1}{|1-z\bar{w}|^{ps}} u(z)\mathrm{d}A(z) \leq \|G_w^s\|_{L^p(u)}^p.$$

To prove the rest conclusions of the lemma, we firstly consider the case when $s > 2p_0/p$. We denote by

$$E_k = S\left(\frac{w}{|w|}, 2^k(1-|w|)\right), \quad k = 0, 1, 2, \ldots,$$

and $\tilde{E}_0 = E_0$, $\tilde{E}_k = E_k \setminus E_{k-1}$, $(k = 1, 2, \ldots)$. It is easy to see that

$$\int_{E_k} \mathrm{d}A(z) \simeq \left(2^k(1-|w|)\right)^2.$$

Then we can obtain the following estimate under this decomposition of $\mathbb{D}$.

- If $z \in \tilde{E}_0$, $|1-z\bar{w}| \geq 1-|w|$, and
- if $z \in \tilde{E}_k$ for $k \geq 1$,

$$|1-z\bar{w}| \geq \left|1 - z\frac{\bar{w}}{|w|}\right| - (1-|w|) \gtrsim 2^k(1-|w|).$$

Since $u \in B_{p_0}$, for every positive integer k, we have

$$\int_{E_k} u(z)\mathrm{d}A(z) \lesssim \frac{A(E_k)^{p_0}}{\left(u^{-p_0'/p_0}\right)(E_k)^{p_0-1}} \leq \frac{A(E_k)^{p_0}}{\left(u^{-p_0'/p_0}\right)(E_0)^{p_0-1}}$$
$$\leq \left(\frac{A(E_k)}{A(E_0)}\right)^{p_0} u(E_0) \lesssim \left(\frac{\left(2^k(1-|w|)\right)^2}{(1-|w|)^2}\right)^{p_0} u(E_0) = 2^{2kp_0} u(E_0).$$

Noting that $s > 2p_0/p$, we can estimate the norm $\|G_w^s\|_{L^p(u)}$ as follows,

$$\begin{aligned}
\|G_w^s\|_{L^p(u)}^p &= \int_{\mathbb{D}} \frac{1}{|1-z\bar{w}|^{ps}} u(z)\mathrm{d}A(z) \\
&= \sum_{k=0}^{\infty} \int_{\tilde{E}_k} \frac{1}{|1-z\bar{w}|^{ps}} u(z)\mathrm{d}A(z) \\
&\lesssim \sum_{k=0}^{\infty} \frac{1}{2^{kps}(1-|w|)^{ps}} \int_{E_k} u(z)\mathrm{d}A(z) \\
&\lesssim \frac{u(E_0)}{(1-|w|)^{ps}} \sum_{k=0}^{\infty} \frac{1}{2^{k(ps-2p_0)}} \lesssim \frac{u(E_0)}{(1-|w|)^{ps}}.
\end{aligned}$$

Now we have proved that (2.3) holds for $s > 2p_0/p$. The case $s = 2p_0/p$ follows from Lemma 3.1 in [4] and also Lemma 2.1 in [5]. So we have

$$\|G_w^{2p_0/p}\|_{A^p(u)} \lesssim \frac{u(S(w))^{\frac{1}{p}}}{(1-|w|)^{2p_0/p}}.$$

When $s < 2p_0/p$, we can see

$$\|G_w^s\|_{A^p(u)} \le 2^{2p_0/p-s}\|G_w^{2p_0/p}\|_{A^p(u)} \simeq \frac{u(S(w))^{\frac{1}{p}}}{(1-|w|)^{2p_0/p}}.$$

That completes the proof. □

The following covering lemma will play a role.

Lemma 2.4 (Theorem 2.23 in [13]) *There exists a positive N such that for any $0 < r \le 1$ we can find a sequence $\{a_k\}$ in $\mathbb{D}$ with the following properties.*

(1) $\mathbb{D} = \cup_k \Delta(a_k, r)$;
(2) *The set $\Delta(a_k, r/4)$ are mutually disjoint;*
(3) *Each point $z \in \mathbb{D}$ belongs to at most N of the sets $\Delta(a_k, 2r)$.*

Any sequence satisfying the conditions in Lemma 2.4 will be called an r-lattice. Note that $|a_k| \to 1^-$as $k \to \infty$. In what follows, the sequence $\{a_k\}$ will always refer to the sequence chosen in Lemma 2.4.

2.1 Carleson Measures

Let $0 < p \le q < \infty$. A positive Borel measure μ on $\mathbb{D}$ is called to be a q-*Carleson measure* for $A^p(u)$ if the embedding $I : A^p(u) \to L^q(\mathrm{d}\mu)$ is bounded. We have the following Carleson embedding theorem.

Lemma 2.5 *Suppose $q \ge p > 0$, $p_0 > 1$ and $0 < r < 1$. Let $u \in B_{p_0}$ be a weight and μ is a positive Borel measure on $\mathbb{D}$. Then the following conditions are equivalent.*

(a) *The embedding $I : A^p(u) \to L^q(\mathrm{d}\mu)$ is bounded, that is*

$$\left(\int_{\mathbb{D}} |f(z)|^q d\mu(z)\right)^{1/q} \lesssim \left(\int_{\mathbb{D}} |f(z)|^p u(z) \mathrm{d}A(z)\right)^{1/p}$$

for every analytic function f on $\mathbb{D}$.
(b) $\mu(S(a)) \lesssim u(S(a))^{q/p}$ *for every $a \in \mathbb{D}$.*
(c) *There is an $r > 0$ such that $\mu(\Delta(a, r)) \lesssim u(\Delta(a, r))^{q/p}$ for every $a \in \mathbb{D}$.*
(d) *There is an $r > 0$ such that $\mu(\Delta(a_k, r)) \lesssim u(\Delta(a_k, r))^{q/p}$ for the sequence $\{a_k\}$ described in Lemma 2.4.*
(e) *Denote by*

$$g_w^s(z) = \frac{1}{u(\Delta(w, r))^{q/p}} \left(\frac{1-|w|^2}{1-z\bar{w}}\right)^s.$$

For any $s \ge 2p_0/p$

$$\|g_w^s\|_{L^q(\mathrm{d}\mu)}^q = \int_{\mathbb{D}} \left|\frac{1-|w|^2}{1-z\bar{w}}\right|^{qs} u(\Delta(w, r))^{-q/p} d\mu(z) \lesssim 1. \tag{2.4}$$

Furthermore, the "geometric norm" of the measure μ, the $L^q(d\mu)$ norm of g_w^s and the operator norm of the embedding are comparable:

$$\sup_{z\in\mathbb{D}}\widehat{\mu_r}(z) := \sup_{z\in\mathbb{D}}\frac{\mu(\Delta(z,r))}{u(\Delta(z,r))^{q/p}} \simeq \sup_{w\in\mathbb{D}}\|g_w^s\|_{L^q(\mathrm{d}\mu)}^q \simeq \|I\|_{A^p(u)\to L^q(\mathrm{d}\mu)}^q.$$

Proof The equivalence **(a)** and **(c)** was proved by Constantin in [5]. We are going to prove **(a)⇒(b)⇒(c)⇒(d)⇒(a)⇒(e)⇒(c)**.

First we prove **(a)⇒(b)**. By choosing a $s \geq 2p_0/p$ we get

$$\frac{\mu(S(a))}{(1-|a|)^{qs}} \lesssim \int_{S(a)} \frac{1}{|1-z\bar{a}|^{qs}}\mathrm{d}\mu(z) \lesssim \|G_a^s\|_{A^p(u)}^q \lesssim \frac{u(S(a))^{q/p}}{(1-|a|)^{qs}},$$

where we use condition **(a)** in the second inequality and Lemma 2.3 in the third inequality.

To prove **(b)⇒(c)**, we let r be sufficiently small and fixed. It will be done to prove $\mu(\Delta(a,r)) \lesssim u(\Delta(a,r))^{q/p}$ for each $|a| \geq \tanh(2r)$. As we state before Lemma 2.1, there is a $a' \in \mathbb{D}$ such that $\Delta(a,r) \subset S\left(a'\right)$ with comparable areas. By Eq. 2.2, we have

$$\mu(\Delta(a,r)) \leq \mu\left(S\left(a'\right)\right) \lesssim u\left(S\left(a'\right)\right)^{q/p} \lesssim u(\Delta(a,r))^{q/p}.$$

The proof of (**c**) ⇒ (**d**) is obvious.

We next prove (**d**) ⇒ (**a**). If f is holomorphic in $\mathbb{D}$, then by Lemma 2.1 we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^q \mathrm{d}\mu(z)\\
&\lesssim \sum_k \int_{\Delta(a_k,r)} \frac{1}{u(\Delta(a_k,r))}\left(\int_{\Delta(a_k,r)} |f(w)|^q u(w)\mathrm{d}A(w)\right)\mathrm{d}\mu(z)\\
&\lesssim \sum_k \int_{\Delta(a_k,r)} \frac{1}{u(\Delta(a_k,r))}\left(\int_{\Delta(a_k,2r)} |f(w)|^q u(w)\mathrm{d}A(w)\right)\mathrm{d}\mu(z)\\
&= \sum_k \frac{\mu(\Delta(a_k,r)}{u(\Delta(a_k,r))}\int_{u(\Delta(a_k,2r))} |f(w)|^q u(w)\mathrm{d}A(w)\\
&\lesssim \sum_k \int_{\Delta(a_k,2r)} u(\Delta(a_k,2r))^{\frac{q-p}{p}} |f(w)|^{q-p}|f(w)|^p u(w)\mathrm{d}A(w)\\
&\lesssim \|f\|_{A^p(u)}^{q-p} \sum_k \int_{\Delta(a_k,2r)} |f(w)|^p u(w)\mathrm{d}A(w)\\
&\lesssim \|f\|_{A^p(u)}^q
\end{aligned}$$

where the last inequality is deduced by Lemma 2.4.

Now we prove **(a)⇒(e)**. Assume that the identity $I : A^p(u) \to L^q(\mathrm{d}\mu)$ is bounded. By Lemma 2.3, we have

$$\|g_w^s\|_{L^q(\mathrm{d}\mu)}^q = \int_{\mathbb{D}} \left|\frac{1-|w|^2}{1-z\bar{w}}\right|^{qs} u(\Delta(w,r))^{-q/p}\mathrm{d}\mu(z) \lesssim \|g_w^s\|_{A^p(u)}^q \simeq 1.$$

To see **(e)⇒(c)**, we assume that (2.4) holds. Then

$$\int_{\Delta(w,r)} \left(\frac{1-|w|^2}{|1-z\bar{w}|}\right)^{qs} u(\Delta(w,r))^{-q/p}\mathrm{d}\mu(z) \lesssim 1.$$

Considering that $\frac{1-|w|^2}{|1-z\bar{w}|} \simeq 1$ when $\rho(z,w) < r$, we find that the left hand side is equivalent to

$$\frac{\mu(\Delta(w,r))}{u(\Delta(w,r))^{q/p}}.$$

That completes the proof. □

2.2 Reproducing Kernels

The key point to prove the main theorems is to estimate the normalized reproducing kernel functions $k_z(w)$ from below. We start our discussion by the following lemma which estimate the reproducing kernel functions on the diagonal.

Lemma 2.6 (Lemma 4.1 in [5]) *Suppose* $p_0 > 1$ *and* $u \in B_{p_0}$. *Let* $K(z,w)$ *be the Bergman kernel in* $A^2(u)$ *and* $r \in (0,1)$. *Then we have the following estimate*

$$K(z,z) \simeq u(\Delta(z,r))^{-1}, \quad z \in \mathbb{D},$$

where the constant involved is independent of $z \in \mathbb{D}$.

Now we can estimate the normalized reproducing kernel $|k_w(z)|$ when z and w are close enough. Our strategy is to update the method of Lemma 3.6 in [8] to our setting.

Theorem 2.7 *Suppose* $p_0 > 1$ *and* $u \in B_{p_0}$. *There is a sufficient small* $\delta \in (0,1)$, *such that*

$$|k_w(z)|^2 \simeq K(z,z)$$

whenever $z \in \Delta(w,\delta)$.

Proof For any fixed $w_0 \in \mathbb{D}$, consider the subspace $A^2(u, w_0)$ of $A^2(u)$, which is defined by

$$A^2(u,w_0) := \left\{ f \in A^2(u) : f(w_0) = 0 \right\}.$$

We have the decomposition

$$A^2(u) = A^2(u,w_0) \oplus \mathcal{L}_{w_0}$$

where $\mathcal{L}_{w_0}$ is the one-dimensional subspace spanned by the function $k_{w_0}(z)$. If we denote by $K_{w_0}(\cdot,\cdot)$ the reproducing kernel of $A^2(u,w_0)$, it is easy to see that

$$K(z,z) = K_{w_0}(z,z) + |k_{w_0}(z)|^2.$$

Hence we have $|k_{w_0}(z)|^2 \leq K(z,z)$. To prove the reverse inequality, we only need to show that there exist constants $0 < C < 1$ and $0 < \delta < 1$, such that

$$K_{w_0}(z,z) \leq CK(z,z) \tag{2.5}$$

whenever $z \in \Delta(w_0,\delta)$. Let us consider the operator

$$\left(S_{w_0} f\right)(z) = \frac{f(z)}{z - w_0}.$$

We claim that S_{w_0} is a bounded mapping from $A^2(u,w_0)$ into $A^2(u)$. Let us see the proof. For every $f \in A^2(u,w_0)$, we have $f(z) = (z-w_0)\tilde{f}(z)$ when $z \in D(w_0,\epsilon)$ for some $\epsilon > 0$ small enough and holomorphic $\tilde{f}$ on $D(w_0, \epsilon)$. It is clear that $S_{w_0} f(z) = \tilde{f}(z)$ whenever $z \in D(w_0, \epsilon)$ and hence it is bounded on $D(w_0, \epsilon)$. Then we have

$$\begin{aligned}\|S_{w_0} f\|_{A^2(u)}^2 &= \int_{\mathbb{D}\setminus D(w_0,\epsilon)} \left|\frac{f(z)}{z-w_0}\right|^2 u(z)\mathrm{d}A(z) + \int_{D(w_0,\epsilon)} |\tilde{f}(z)|^2 u(z)\mathrm{d}A(z) \\ &\lesssim \int_{\mathbb{D}\setminus D(w_0,\epsilon)} \frac{|f(z)|^2}{\epsilon^2} u(z)\mathrm{d}A(z) + \pi\epsilon^2 \\ &\leq \frac{1}{\epsilon^2}\|f\|_{A^2(u)}^2 + \pi\epsilon^2 < \infty.\end{aligned}$$

That means $S_{w_0} f \in A^2(u)$ for every $f \in A^2(u, w_0)$.

Define the $V_{w_0}^z : \mathbb{C} \to \mathbb{C}$ by $V_{w_0}^z(\xi) = (z - w_0)\xi$. Then the point evaluation $U_{w_0}^z f = f(z)$ on $A^2(u, w_0)$ can be represented as

$$U_{w_0}^z = V_{w_0}^z L_z S_{w_0}$$

where L_z is the point evaluation on $A^2(u)$. Hence

$$\|U_{w_0}^z\|_{A^2(u,w_0)\to\mathbb{C}} \leq \|V_{w_0}^z\|_{\mathbb{C}\to\mathbb{C}} \|L_z\|_{A^2(u)\to\mathbb{C}} \|S_{w_0}\|_{A^2(u,w_0)\to A^2(u)}.$$

Note that $\|V_{w_0}^z\|_{\mathbb{C}\to\mathbb{C}} = |z - w_0|$. To estimate the norm of S_{w_0} for any $f \in A^2(u, w_0)$, let $g(z) = f(z)(z - w_0)^{-1} = (S_{w_0} f)(z)$. Then $g \in A^2(u)$, since S_{w_0} maps $A^2(u, w_0)$ into $A^2(u)$. According to Lemma 2.6, we fix a $r \in (0, 1)$ so that there is a constant C independent on the choice of $z \in \mathbb{D}$ with $K(z, z) \leq Cu(\Delta(z, r))^{-1}$. Hence we have

$$\|g\|_{A^2(u)}^2 = \int_{\mathbb{D}} |g|^2 u\mathrm{d}A = \left(\int_{\Delta(w_0,\frac{r}{k})} + \int_{\mathbb{D}\setminus\Delta(w_0,\frac{r}{k})}\right) |g|^2 u\mathrm{d}A = \mathrm{I}_k + \mathrm{II}_k,$$

where k is an integer. By the reproducing property we have

$$g(z) = \int_{\mathbb{D}} K(z, w) g(w) u(w)\mathrm{d}A(w)$$

It follows that

$$\begin{aligned}\mathrm{I}_k &= \int_{\Delta(w_0,r/k)} |g(z)|^2 u(z)\mathrm{d}A(z) \\ &\leq \int_{\Delta(w_0,r/k)} \|K(\cdot, z)\|_{A^2(u)}^2 \|g\|_{A^2(u)}^2 u(z)\mathrm{d}A(z) \\ &= \int_{\Delta(w_0,r/k)} K(z, z) u(z)\mathrm{d}A(z) \cdot \|g\|_{A^2(u)}^2.\end{aligned}$$

By Lemma 2.6, we obtain

$$\begin{aligned}&\int_{\Delta(w_0,r/k)} K(z, z) u(z)\mathrm{d}A(z) \leq C \int_{\Delta(w_0,r/k)} u(\Delta(z, r))^{-1} u(z)\mathrm{d}A(z) \\ &\leq C' \frac{1}{u(\Delta(w_0, r))} \int_{\Delta(w_0,\frac{r}{k})} u(z)\mathrm{d}A(z) \\ &= C' \frac{u\left(\Delta\left(w_0, \frac{r}{k}\right)\right)}{u(\Delta(w_0, r))},\end{aligned}$$

which converges to 0 as k goes to infinity. Combining this fact with

$$\|g\|^2_{A^2(u)} = \mathrm{I}_k + \mathrm{II}_k,$$

now we can choose a k large enough such that

$$\begin{aligned}
\|g\|^2_{A^2(u)} &\leq C'' \int_{\mathbb{D}\setminus\Delta(w_0,\frac{r}{k})} |g(z)|^2 u(z) \mathrm{d}A(z) \\
&= C'' \int_{\mathbb{D}\setminus\Delta(w_0,\frac{r}{k})} \left|\frac{f(z)}{z-w_0}\right|^2 u(z) \mathrm{d}A(z) \\
&= C'' \int_{\mathbb{D}\setminus\Delta(w_0,\frac{r}{k})} \left|\frac{1-\overline{w_0}z}{z-w_0}\right|^2 \frac{1}{|1-\overline{w_0}z|^2} |f(z)|^2 u(z) \mathrm{d}A(z) \\
&\leq \left(\frac{k}{r}\right)^2 \frac{C''}{(1-|w_0|)^2} \int_{\mathbb{D}\setminus\Delta(w_0,\frac{r}{k})} |f(z)|^2 u(z) \mathrm{d}A(z) \\
&\leq \frac{C''\,(k/r)^2}{(1-|w_0|)^2} \|f\|^2_{A^2(u)}.
\end{aligned}$$

It then follows that

$$\|S_{w_0}\|_{A^2(u,w_0)\to A^2(u)} \leq \frac{\sqrt{C''}(k/r)}{1-|w_0|},$$

where C'' is independent on w_0, k is an integer and $r \in (0, 1)$ is fixed. Hence

$$\begin{aligned}
\|U^z_{w_0}\|_{A^2(u,w_0)\to\mathbb{C}} &\leq \|V^z_{w_0}\|_{\mathbb{C}\to\mathbb{C}} \|L_z\|_{A^2(u)\to\mathbb{C}} \|S_{w_0}\|_{A^2(u,w_0)\to A^2(u)} \\
&\leq \frac{\sqrt{C''}(k/r)|z-w_0|}{1-|w_0|} \|L_z\|_{A^2(u)\to\mathbb{C}}
\end{aligned}$$

Since $U^z_{w_0}$ and L_z are point evaluations on $A^2(u, w_0)$ and $A^2(u)$ respectively, by the Riesz representation we have

$$\|U^z_{w_0}\|^2_{A^2(u,w_0)\to\mathbb{C}} = K_{w_0}(z, z) \quad \text{and} \quad \|L_z\|^2_{A^2(u)\to\mathbb{C}} = K(z, z).$$

We let $\rho(z, w_0) < \delta$ where $\delta \in (0, r)$ will be specified later. We obtain that

$$\begin{aligned}
K_{w_0}(z, z) &\leq \frac{k\sqrt{C''}}{r} \frac{|1-\overline{w_0}z|}{1-|w_0|} \rho(z, w_0) K(z, z) \\
&\leq \delta \frac{k\sqrt{C''}}{r} \cdot \frac{|1-\overline{w_0}z|}{1-|w_0|} K(z, z) \leq \delta \frac{C'''k}{r} K(z, z)
\end{aligned}$$

where C''' is independent on w_0 and z by Eq. 2.1. We now choose $\delta > 0$ such that $\left(C'''k/r\right) \cdot \delta < 1$, and this completes the proof of (2.5) and of the theorem. □

The following proposition is proved by Chacón in [3] which is going to be employed in the proof of the compactness.

Proposition 2.8 *Let $p_0 > 1$. If $u \in B_{p_0}$, then the normalized kernel function k_w converges to zero weakly in $A^2(u)$.*

The next Proposition is a classical result, its proof is similar to that one given by K. Zhu in [14, Theorem1.14].

Proposition 2.9 *Let $p_0 > 1$ and $u \in B_{p_0}$. A linear operator T on $A^2(u)$ is compact if and only if $\|Tf_n\|_{A^2(u)} \longrightarrow 0$ whenever $f_n \to 0$ weakly in $A^2(u)$.*

3 Proof of Theorem 1.1

Proof The equivalence **(i)⇔(iii)** was proved by Constantin in [5] Theorem 4.1.

We prove "**(ii)⇒(iii)**" first. Since $0 < r \le \delta$, we use Theorem 2.7, Lemmas 2.6 and 2.2 to see that

$$\begin{aligned}\tilde{\mu}(z) &\ge \int_{\Delta(z,r)} |k_z(\zeta)|^2 \mathrm{d}\mu(\zeta) \simeq \int_{\Delta(z,r)} K(\zeta,\zeta)\mathrm{d}\mu(\zeta)\\ &\simeq \int_{\Delta(z,r)} \frac{1}{u(\Delta(\zeta,r))}\mathrm{d}\mu(\zeta) \simeq \frac{1}{u(\Delta(z,r))}\int_{\Delta(z,r)} \mathrm{d}\mu(\zeta) = \widehat{\mu_r}(z).\end{aligned} \tag{3.1}$$

To prove "**(iii)⇒(ii)**, by Lemma 2.1 one has that

$$|k_z(w)|^2 \lesssim \frac{1}{u(\Delta(w,r))}\int_{\Delta(w,r)} |k_z(\zeta)|^2 u(\zeta)\mathrm{d}A(\zeta),$$

where $z \in \mathbb{D}$ and $r > 0$. Let $\{a_j\}$ and $r > 0$ be chosen as in Lemma 2.4. We can use Lemma 2.2, Fubini's Theorem and Lemma 2.4 to conduct the following computation

$$\begin{aligned}\tilde{\mu}(z) &\le \sum_{j=1}^{\infty} \int_{\Delta(a_j,r)} |k_z(w)|^2 \mathrm{d}\mu(w)\\ &\lesssim \sum_{j=1}^{\infty} \int_{\Delta(a_j,r)} \frac{1}{u(\Delta(w,r))}\int_{\Delta(w,r)} |k_z(\zeta)|^2 u(\zeta)\mathrm{d}A(\zeta)\mathrm{d}\mu(w)\\ &\lesssim \sup_j \frac{\mu(\Delta(a_j,r))}{u(\Delta(a_j,r))}\sum_{j=1}^{\infty}\int_{\Delta(a_j,2r)} |k_z(\zeta)|^2 u(\zeta)\mathrm{d}A(\zeta)\\ &\lesssim \sup_{z\in\mathbb{D}} \widehat{\mu_r}(z),\end{aligned}$$

where the last inequality follows the fact that **(3)** in Lemma 2.4. That completes the proof. □

4 Proof of Theorem 1.2

Proof Let $I : A_b^2(u) \to L^2(\mathrm{d}\mu)$ be the identity. According to the observation

$$T_\mu = I^* I$$

and the vanishing Carleson embedding theorem (Theorem 3.3 in [5]), we have that T_μ is compact on $A^2(u)$ if and only if $\lim\limits_{|z|\to 1} \widehat{\mu_r}(z) = 0$. Hence we have the equivalence "**(i)⇔(iii)**". "**(ii)⇒(iii)**" is an obvious consequence of the inequality (3.1).

To prove "**(iii)**$\Rightarrow$**(ii)**", assume that $\lim_{|z|\to 1} \widehat{\mu_r}(z) = 0$. Let $\{a_n\}$ and $r \in (0, \delta]$ be chosen as in Lemma 2.4. For any $\epsilon > 0$, let N be the integer that $\widehat{\mu_r}(a_n) < \epsilon$ whenever $n \geq N$. We can find a compact subset $K \subset \mathbb{D}$ such that $K \supset \cup_{j=1}^{N} \Delta\left(a_j, r\right)$. Then we have that

$$\tilde{\mu}(z) \leq \int_K |k_z(w)|^2 \mathrm{d}\mu(w) + \sum_{j>N} \int_{\Delta(a_j,r)} |k_z(w)|^2 \mathrm{d}\mu(w) := \mathrm{I}(z) + \mathrm{II}(z).$$

Since **(iii)** means that T_μ is compact, Proposition 2.8 implies that $\lim_{|z|\to 1} \mathrm{I}(z) = 0$. To complete the proof, we estimate $\mathrm{II}(z)$ as follows: by Lemmas 2.1 and 2.4, we have

$$\begin{aligned}
\mathrm{II}(z) &\lesssim \sum_{j>N} \int_{\Delta(a_j,r)} \frac{1}{u(\Delta(w,r))} \int_{\Delta(w,r)} |k_z(\zeta)|^2 u(\zeta) \mathrm{d}A(\zeta) \mathrm{d}\mu(w) \\
&\lesssim \sup_{j>N} \frac{\mu(\Delta(a_j,r))}{u(\Delta(a_j,r))} \sum_{j>N} \int_{\Delta(a_j,2r)} |k_z(\zeta)|^2 u(\zeta) \mathrm{d}A(\zeta) \\
&\lesssim \sup_{j>N} \widehat{\mu_r}(a_j) < \epsilon,
\end{aligned}$$

which gives the desired result. □

5 Proof of Theorem 1.3

Let T be a compact operator and $h : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous increasing function. The authors of [6] introduce the following class of operators on a Hilbert space H. Say that $T \in \mathcal{S}_h(H)$ if there is a positive constant $c > 0$ such that

$$\sum_{n=1}^{\infty} h(cs_n(T)) < \infty$$

where $s_n(T)$ is the nth singular value of T.

To study the Schatten class of the Toeplitz operators on the Bergman spaces with Békollé-Bonami weights, we follows the strategy of [6]. The following Lemma is the generalized version of Theorem 6.2 in [6].

Lemma 5.1 *Let $p_0 > 1$, $u \in B_{p_0}$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ an increasing convex function. Let μ be a positive Borel measure on $\mathbb{D}$ such that the Toeplitz operator T_μ is compact on $A^2(u)$. Then $T_\mu \in \mathcal{S}_h\left(A^2(u)\right)$ if and only if there is a constant $C > 0$ such that $\int_{\mathbb{D}} h\left(C\tilde{\mu}(z)\right) d\lambda_u(z) < \infty$ where $d\lambda_u(z) = \|K_z\|^2 u(z) \mathrm{d}A(z)$.*

Proof Assume that $T_\mu \in \mathcal{S}_h\left(A^2(u)\right)$. That is $\sum_n h(Cs_n) < \infty$ for some positive constant C. Let $\{e_n\}$ be an orthonormal set in $A^2(u)$ and $T_\mu = \sum_{n=1}^{\infty} s_n \langle \cdot, e_n \rangle e_n$ the canonical decomposition of the positive operator T_μ where s_n are also the eigenvalues of T_μ. Note that k_z is the normalized reproducing kernel functions in $A^2(u)$. We have $\sum_n |\langle k_z, e_n\rangle|^2 = 1$ by the Parseval formula. Then it follows by the convexity of h and Jensen's inequality that

$$\begin{aligned}\int_{\mathbb{D}} h\left(C\tilde{\mu}(z)\right) d\lambda_u(z) &= \int_{\mathbb{D}} h\left(C\langle T_\mu k_z, k_z\rangle_{A^2(u)}\right) \mathrm{d}\lambda_u(z)\\ &= \int_{\mathbb{D}} h\left(\sum_{n=1}^{\infty} Cs_n|\langle k_z, e_n\rangle_{A^2(u)}|^2\right) \mathrm{d}\lambda_u(z)\\ &\le \int_{\mathbb{D}} \sum_n h(Cs_n)|\langle k_z, e_n\rangle_{A^2(u)}|^2 \mathrm{d}\lambda_u(z)\\ &= \int_{\mathbb{D}} \sum_n h(Cs_n)\|K(\cdot, z)\|^{-2}|e_n(z)|^2 d\lambda_u(z)\\ &= \sum_n h(Cs_n)\int_{\mathbb{D}} |e_n(z)|^2 u(z) dA(z) = \sum_n h(Cs_n) < \infty.\end{aligned}$$

Conversely, we assume that $\int_{\mathbb{D}} h(C\tilde{\mu}(z))\mathrm{d}\lambda_u(z) < \infty$ for some $C > 0$. By Lemmas 2.1, 2.4, 2.6 and Theorem 2.7, we obtain

$$\begin{aligned}\langle T_\mu e_n, e_n\rangle &= \int_{\mathbb{D}} |e_n(z)|^2 d\mu(z)\\ &\lesssim \int_{\mathbb{D}} (u(\Delta(z,r))^{-1} \int_{\Delta(z,\delta)} |e_n(\xi)|^2 u(\xi) dA(\xi) d\mu(z)\\ &\simeq \int_{\mathbb{D}} \|K(\cdot, z)\|^2 \int_{\Delta(z,\delta)} |e_n(\xi)|^2 u(\xi) dA(\xi) d\mu(z)\end{aligned}$$

By Fubini's theorem, Lemma 2.2 and Theorem 2.7, we get

$$\begin{aligned}\langle T_\mu e_n, e_n\rangle &= \int_{\mathbb{D}} \left(\int_{\Delta(\xi,\delta)} K(z,z)\mathrm{d}\mu(z)\right) |e_n(\xi)|^2 u(\xi) dA(\xi)\\ &\lesssim \int_{\mathbb{D}} \left(\int_{\Delta(\xi,\delta)} |k_\xi(z)|^2 d\mu(z)|\right) e_n(\xi)|^2 u(\xi) dA(\xi)\\ &\le \int_{\mathbb{D}} \tilde{\mu}(\xi)|e_n(\xi)|^2 u(\xi) dA(\xi).\end{aligned}$$

It then follows by Jensen's formula that

$$\begin{aligned}\sum_{n=1}^{\infty} h(C_1\langle T_\mu e_n, e_n\rangle) &\lesssim \int_{\mathbb{D}} h(C_1\tilde{\mu}(\xi)) \left(\sum_{n\ge 1} |e_n(\xi)|^2\right) u(\xi) dA(\xi)\\ &= \int_{\mathbb{D}} h(C_1\tilde{\mu}(\xi))\|K(\cdot, \xi)\|^2 u(\xi) dA(\xi)\\ &= \int_{\mathbb{D}} h(C_1\tilde{\mu}(\xi)) d\lambda_u(\xi) < \infty.\end{aligned}$$

Therefore $T_\mu \in \mathcal{S}_h(A^2(u))$. □

As a direct consequence of Lemma 5.1, we give the proof of Theorem 1.3 which is the Schatten class of the Toeplitz operators on $A^2(u)$.

Proof of Theorem 1.3 Let $h(t) = t^p$ where $p > 1$ and use Lemma 5.1. □

Another application of Lemma 5.1 is on the decay of the eigenvalue of T_μ which is regarded as a generalization of Theorem 6.4 in [6].

Corollary 5.2 *Let $p_0 > 1$ and $u \in B_{p_0}$. A continuous decreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\eta(t) \to 0$ and $\eta(t) \simeq \eta(t \log t)$ as $t \to \infty$. Let h_η be the function that $h_\eta(\eta(t)) = 1/t$. Then $s_n(T_\mu) = O(\eta(n))$ if and only if*

$$\int_{\mathbb{D}} h_\eta(C\tilde{\mu}(z))\mathrm{d}\lambda_u(z) < \infty$$

for some positive constant C.

Proof Use Lemma 6.1 in [6] and Theorem 5.1. □

6 Proof of Theorem 1.4

In the proof of Theorem 1.1 we obtain $\widehat{\mu_r}(z) \le \tilde{\mu}(z)$, for $z \in \mathbb{D}$ and $r \in (0, \delta)$. Therefore, it is enough to prove

$$\limsup_{|z|\to 1^-} \tilde{\mu}(z) \lesssim \|T_\mu\|_e \lesssim \limsup_{|z|\to 1^-} \widehat{\mu_r}(z).$$

Firstly, we start with the lower estimate. We take an arbitrary compact operator K on $A^2(u)$. By Proposition 2.8, the normalized reproducing kernel $\{k_z\}$ converges to 0 weakly in $A^2(u)$. Then, $\|Kk_z\|_{A^2(u)} \to 0$ as $|z| \to 1^-$, by Proposition 2.9. Therefore,

$$\|T_\mu - K\| \ge \limsup_{|z|\to 1^-} \|(T_\mu - K)k_z\|_{A^2(u)} \ge \limsup_{|z|\to 1^-} \|T_\mu k_z\|_{A^2(u)}. \tag{6.1}$$

Since (6.1) holds for any compact operator K, it follows that

$$\|T_\mu\|_e \ge \limsup_{|z|\to 1^-} \|T_\mu k_z\|_{A^2(u)}. \tag{6.2}$$

On the other hand, since T_μ is bounded, we have

$$\tilde{\mu}(z) = |\langle T_\mu k_z, k_z\rangle_{A^2(u)}| \le \|T_\mu k_z\|_{A^2(u)}.$$

Combining this with (6.2), we get the lower estimate.

Now we prove the upper estimate for the essential norm of Toeplitz operators T_μ. Suppose $\{e_n\}$ is a complete orthonormal system of $A^2(u)$. For $n \in \mathbb{N}$, we define an operator Q_n by

$$Q_n f := \sum_{j=1}^{n} \langle f, e_j\rangle_{A^2(u)} e_j, \quad \text{for any} \quad f \in A^2(u).$$

The operators Q_n is compact on $A^2(u)$. Let $R_n = I - Q_n$. It is easy to see that $R_n^* = R_n$ and $R_n^2 = R_n$. Furthermore, we have

$$\lim_{n\to+\infty} \|R_n f\|_{A^2(u)} = 0, \quad \text{for any} \quad f \in A^2(u).$$

For $\rho > 0$, let $D_\rho = \mathbb{D} \setminus D(0, \rho)$ and $d\mu_\rho(z) = \chi_{D_\rho}(z)d\mu(z)$, where χ_{D_ρ} is the characteristic function on D_ρ. By the definition of the average function of μ_ρ, we can see

$$\widehat{[\mu_\rho]}_r(z) = \frac{1}{u(\Delta(z,r))} \int_{\Delta(z,r)\cap D_\rho} d\mu(\xi) = \frac{1}{u(\Delta(z,r))} \int_{\Delta(z,r)\cap D_\rho} d\mu(\xi).$$

To finish this proof we need two following lemmas that we are going to prove later at the end of this paper. The first lemma is the identity (1.1) that we mentioned in the beginning.

Lemma 6.1 *Let μ be a positive measure on $\mathbb{D}$ and $u \in B_{p_0}$, with $p_0 > 1$. Suppose T_μ is bounded on $A^2(u)$. Then*

$$\langle T_\mu f, g\rangle_{A^2(u)} = \int_{\mathbb{D}} f(\xi)\,\overline{g(\xi)}\,d\mu(\xi), \qquad f, g \in A^2(u).$$

Proof Fubini's theorem and reproducing kernel formula give

$$\begin{aligned}\langle T_\mu f, g\rangle_{A^2(u)} &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(\xi)\,\overline{K_z(\xi)}\,d\mu(\xi)\right) \overline{g(z)}\,u(z)\,dA(z)\\ &= \int_{\mathbb{D}} f(\xi) \left(\int_{\mathbb{D}} \overline{g(z)}\,K_\xi(z)\,u(z)\,dA(z)\right) d\mu(\xi)\\ &= \int_{\mathbb{D}} f(\xi)\,\overline{\langle g(z), K_\xi\rangle_{A^2(u)}}\,d\mu(\xi)\\ &= \int_{\mathbb{D}} f(\xi)\,\overline{g(\xi)}\,d\mu(\xi).\end{aligned}$$

This finishes the proof. □

Lemma 6.2 *Suppose T_μ is bounded on $A^2(u)$. For any $\rho > 0$ and $r \in (0, \delta)$, one has*

$$\lim_{n\to+\infty} \sup_{\|f\|_{A^2(u)}=1} \|T_\mu R_n f\|_{L^2(d\mu)} \lesssim \sup_{z\in\mathbb{D}} \widehat{\left[\mu_\rho\right]_r}(z) \tag{6.3}$$

and

$$\lim_{n\to+\infty} \sup_{\|f\|_{A^2(u)}=1} \|R_n f\|_{L^2(d\mu)} \lesssim \sup_{z\in\mathbb{D}} \widehat{\left[\mu_\rho\right]_r}(z). \tag{6.4}$$

Lemma 6.3 *Suppose T_μ is bounded on $A^2(u)$. For any $r \in (0, \delta)$ and $\rho > r$, one has*

$$\sup_{z\in\mathbb{D}} \widehat{\left[\mu_\rho\right]_r}(z) \lesssim \sup_{z\in D_{\rho-r}} \widehat{\mu_r}(z).$$

Assume that all results given by Lemmas 6.2 and 6.3 are true. Since Q_n is compact, $T_\mu Q_n$ is also compact. Therefore, we have

$$\|T_\mu\|_e \le \|T_\mu - T_\mu Q_n\| = \|T_\mu R_n\|.$$

It is easy to see that

$$\|T_\mu R_n f\|^2_{A^2(u)} \le \|R_n f\|_{L^2(d\mu)}\, \|T_\mu R_n f\|_{L^2(d\mu)}.$$

Moreover, we have

$$\|T_\mu\|_e^2 \le \|T_\mu R_n\|^2 \le \sup_{\|f\|_{A^2(u)}=1} \|R_n f\|_{L^2(d\mu)} \sup_{\|f\|_{A^2(u)}=1} \|T_\mu R_n f\|_{L^2(d\mu)}.$$

Taking $n \to \infty$, by Lemmas 6.2 and 6.3, we have

$$\|T_\mu\|_e^2 \le \left(\sup_{z\in\mathbb{D}} \widehat{\left[\mu_\rho\right]_r}(z)\right)^2 \lesssim \left(\sup_{z\in D_{\rho-r}} \widehat{\mu_r}(z)\right)^2.$$

Letting $\rho \to r+1$, we obtain

$$\|T_\mu\|_e \lesssim \limsup_{|z|\to 1^-} \widehat{\mu_r}(z).$$

This completes the proof.

6.1 Proof of Lemma 6.2

Since the proofs of (6.3) and (6.4) are almost the same, we only prove (6.3). First, we show

$$\lim_{n\to+\infty} \sup_{\|f\|_{A^2(u)}=1} \int_{D(0,\rho)} |T_\mu R_n f(z)|^2 \, d\mu(z) = 0. \tag{6.5}$$

Since $T_\mu R_n f \in A^2(u)$, we obtain

$$|T_\mu R_n f(z)| = |\langle T_\mu R_n f, K_z\rangle_{A^2(u)}| = |\langle f, R_n T_\mu^* K_z\rangle_{A^2(u)}| \le \|f\|_{A^2(u)} \|R_n T_\mu^* K_z\|_{A^2(u)},$$

where the first equality follows from the reproducing property. Then, we get

$$\sup_{\|f\|_{A^2(u)}=1} \int_{D(0,\rho)} |T_\mu R_n f(z)|^2 \, d\mu(z) \le \int_{D(0,\rho)} \|R_n T_\mu^* K_z\|^2_{A^2(u)} \, d\mu(z).$$

Therefore, it is enough to prove

$$\lim_{n\to\infty} \int_{D(0,\rho)} \|R_n T_\mu^* K_z\|^2_{A^2(u)} \, d\mu(z) = 0.$$

This follows from Lebesgue's dominated convergence theorem because of

$$\|R_n T_\mu^* K_z\|^2 \le \|T_\mu\|^2_{A^2(u)} \|K_z\|^2_{A^2(u)} = \|T_\mu\|^2_{A^2(u)} K(z,z)$$

and $K(z,z) \in L^\infty(D(0,\rho), d\mu)$.

Next, we prove

$$\sup_{\|f\|_{A^2(u)}=1} \int_{D_\rho} |T_\mu R_n f(z)|^2 \, d\mu(z) \lesssim \sup_{z\in\mathbb{D}} \widehat{[\mu_\rho]_r}(z). \tag{6.6}$$

This comes from

$$\begin{aligned} \int_{D_\rho} |T_\mu R_n f(z)|^2 \, d\mu(z) &= \int_{\mathbb{D}} |T_\mu R_n f(z)|^2 \, d\mu_\rho(z) \\ &\lesssim \int_{\mathbb{D}} \widehat{[\mu_\rho]_r}(z) \, |T_\mu R_n f(z)|^2 u(z) \, dA(z) \\ &\lesssim \sup_{z\in\mathbb{D}} \widehat{[\mu_\rho]_r}(z) \|T_\mu R_n f\|^2_{A^2(u)} \\ &\lesssim \|T_\mu\|^2_{A^2(u)} \|f\|^2_{A^2(u)} \sup_{z\in\mathbb{D}} \widehat{[\mu_\rho]_r}(z). \end{aligned}$$

From Eqs. 6.6 and 6.5 we obtain 6.3 and complete the proof.

6.2 Proof of Lemma 6.3

By definition of the averaging,

$$\widehat{[\mu_\rho]}_r(z) = \frac{1}{u(\Delta(z,r))}\int_{\Delta(z,r)} d\mu_\rho(\xi) = \frac{1}{u(\Delta(z,r))}\int_{\Delta(z,r)\cap D_\rho} d\mu(\xi)$$

By Theorem 2.7, we obtain

$$\widehat{[\mu_\rho]}_r(z) \lesssim \int_{\Delta(z,r)\cap D_\rho} |k_z(\xi)|^2\, d\mu(\xi). \tag{6.7}$$

By Lemma 2.1, we have

$$|k_z(\xi)|^2 \lesssim \frac{1}{u(\Delta(\xi,r))}\int_{\Delta(\xi,r)} |k_z(s)|^2\, u(s)\, dA(s),$$

for any $\xi \in \Delta(z,r)$. Plugging this into Eq. 6.7, by Lemma 2.2 and Fubini's theorem, we have

$$\begin{aligned}
\widehat{[\mu_\rho]}_r(z) &\lesssim \int_{\Delta(z,r)\cap D_\rho}\int_{\mathbb{D}} \chi_{\Delta(\xi,r)}(s)\, |k_z(s)|^2\, u(s)\, dA(s)\, d\mu(\xi) \\
&\lesssim \int_{\mathbb{D}}\left(\int_{\Delta(z,r)\cap D_\rho} \frac{\chi_{\Delta(s,r)}(\xi)}{u(\Delta(\xi,r))}\, d\mu(\xi)\right) |k_z(s)|^2\, u(s)\, dA(s) \\
&\le \sup_{s\in\mathbb{D}}\left(\int_{\Delta(z,r)\cap D_\rho} \frac{\chi_{\Delta(s,r)}(\xi)}{u(\Delta(\xi,r))}\, d\mu(\xi)\right)\int_{\mathbb{D}} |k_z(s)|^2\, u(s)\, dA(s) \\
&= \sup_{s\in\mathbb{D}}\left(\int_{\Delta(z,r)\cap\Delta(s,r)\cap D_\rho} \frac{1}{u(\Delta(\xi,r))}\, d\mu(\xi)\right).
\end{aligned}$$

Now we show that $\Delta(z,r)\cap\Delta(s,r)\cap D_\rho = \emptyset$, for any $s \in D(0,\rho-r)$. Indeed, if $\xi \in \Delta(z,r)\cap\Delta(s,r)\cap D_\rho$, we have $d(\xi,s) \le r$ and $d(0,\xi) \ge \rho$, so that

$$d(0,s) \ge d(0,\xi) - d(\xi,s) \ge \rho - r,$$

which means that $s \notin D(0,\rho-r)$. Therefore,

$$\begin{aligned}
\widehat{[\mu_\rho]}_r(z) &\lesssim \sup_{s\notin D(0,\rho-r)}\left(\int_{\Delta(z,r)\cap\Delta(s,r)\cap D_\rho} \frac{1}{u(\Delta(\xi,r))}\, d\mu(\xi)\right) \\
&\lesssim \sup_{s\in D_{\rho-r}} \frac{1}{u(\Delta(s,r))}\left(\int_{\Delta(z,r)\cap\Delta(s,r)\cap D_\rho} d\mu(\xi)\right) \\
&\lesssim \sup_{s\in D_{\rho-r}} \widehat{\mu_r}(s),
\end{aligned}$$

which completes the proof.

References

1. Bekollé, D.: Inégalité à poids pour le projecteur de Bergman dans la boule unité de $\mathbb{C}^n$. Studia Math. **71**(3), 305–323 (1981/1982). French
2. Bekollè, D., Bonami, A.: Inégalités à poids pour le noyau de Bergman. C. R. Acad. Sci. Paris Sér. A-B **286**(18), A775–A778 (1978). French
3. Chacón, G.R.: Toeplitz operators on weighted Bergman spaces. J. Funct. Space. Appl. **2013**(753153), 5 (2013)
4. Constantin, O.: Discretizations of integral operators and atomic decompositions in vector-valued weighted Bergman spaces. Integr. Equ. Oper. Theory **523-554**, 59 (2007)
5. Constantin, O.: Carleson embeddings some classes of operators on weighted Bergman spaces. J. Math. Anal. Appl. **365**, 668–682 (2010)
6. El-Fallah, O., Mahzouli, H., Marrhich, I., Naqos, H.: Asymptotic behavior of eigenvalues of Toeplitz operators on the weighted analytic spaces. J. Funct. Anal. **270**, 4614–4630 (2016)
7. Luecking, D.: Representation and duality in weighted spaces of analytic functions. Indiana Univ. Math. J. **34**, 319–336 (1985)
8. Lin, P., Rochberg, R.: Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights. Pac. J. Math. **127-146**, 173 (1996)
9. Pott, S., Reguera, M.C.: Sharp Bekolle estimates for the Bergman projection. J. Funct. Anal. **265**(12), 3233–3244 (2013)
10. Rahm, R., Tchoundja, E., Wick, B.D.: Weighted estimates for the Berezin transform and Bergman projection on the unit ball in $\mathbb{C}^n$. Math. Z. **286**(3), 1465–1478 (2017). to appear
11. Tong, C., Li, J.: Carleson measures on the weighted Bergman spaces with Békollé weights, Chinese Ann. Math. Ser. B, accepted
12. Tong, C., Yuan, C.: An integral operator preserving s-Carleson measure on the unit ball. Ann. Acad. Sci. Fen. Math. **40**, 361–373 (2015)
13. Zhu, K.: Spaces of Holomorphic Functions in the Unit Ball. Grad Texts in Math. Springer, New York (2005)
14. Zhu, K.: Operator Theory in Function Spaces, Second Edition, Math Surveys and Monographs 138. American Mathematical Society, Providence, Rhode Island (2007)