A Lie symmetry analysis and explicit solutions of the two-dimensional ∞-Polylaplacian

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A LIE SYMMETRY ANALYSIS AND EXPLICIT SOLUTIONS OF THE TWO DIMENSIONAL ∞-POLYLAPLACIAN

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ABSTRACT. In this work we consider the Lie point symmetry analysis of a strongly nonlinear partial differential equation of third order, the ∞-Polylaplacian, in two spatial dimensions. This equation is a higher order generalisation of the ∞-Laplacian, also known as Aronsson's equation, and arises as the analogue of the Euler-Lagrange equations of a second order variational principle in L^∞. We obtain its full symmetry group, one dimensional Lie sub-algebras and the corresponding symmetry reductions to ordinary differential equations. Finally, we use the Lie symmetries to construct new invariant ∞-Polyharmonic functions.

1. INTRODUCTION

In recent years many partial differential equations (PDEs) that appear as Euler-Lagrange equations in L^∞ variational problems have drawn considerable attention, see [BEJ08, Bar99, EY05, Kat15] and references therein. These equations are strongly nonlinear elliptic PDEs and appear in many important applications such as modes for travelling waves in suspension bridges [GM10, LM90], the modelling of granular matter [Igb12], image processing [ETT15] and game theory [BEJ08].

In this work we study the ∞-Polylaplacian equation

$$(1.1) \qquad \Pi^2_\infty u := \sum_{i,j=1}^n f[u]_{x_i} f[u]_{x_j} u_{x_i x_j} = 0,$$

where $u = u(x_1, \ldots, x_n) \in \mathbb{R}$ and $f[u]$ is given by

$$(1.2) \qquad f[u] := \sum_{i,j=1}^n (u_{x_i x_j})^2,$$

from a Lie-algebraic and computational point of view. As usual the lower index denotes partial differentiation with respect to the corresponding variable. It is surprising that equation (1.1) is a *third order* PDE since it is the formal limit of the p-Polylaplacian

$$(1.3) \qquad \sum_{i,j=1}^n \left(f[u]^{p/2-1} u_{x_i x_j} \right)_{x_i x_j} = 0$$

as $p \to \infty$ which is a *fourth order* PDE. Moreover, the ∞-Polylaplacian (1.1) has a connection to an equation that can be seen as a higher order generalisation of the well known Eikonal equation. Indeed, given the structure of the operator (1.1), it is clear that solutions to the second order Eikonal-type equation

$$(1.4) \qquad f[u] = c,$$

where c is a constant, are also solutions to the ∞-Polylaplacian.

Equation (1.1) can be seen as a higher order generalisation of the ∞-Laplacian equation [Aro65]

$$(1.5) \qquad \Delta_\infty u := \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0,$$

also known as Aronsson's equation. The symmetries of equation (1.5) with $n = 2$ were recently studied and exact solutions were constructed in [FF11], see also [Aya18]. Aronsson's equation minimises the Dirichlet energy functional in L^p as $p \to \infty$, see also [KP16, Pry17] for a modern review of the derivation. There are many difficulties typically encountered in these variational problems and the study of the associated Euler-Lagrange equations obtained in this way are notoriously challenging [Kat15]. Usually solutions are non-classical and need to be made sense of weakly. The correct notion of weak solutions in this context is that of *viscosity solutions* [BDM89, Jen93]. In the context of the ∞-Polylaplacian the notion of viscosity solutions is no longer applicable since we do not have access to a maximum principle for 3rd order PDEs, from which the solution concept stems. It is also difficult, due to their complicated form, to construct exact and physically interesting solutions. In [KP17] equations of this type and the structure of their solutions were studied using appropriate numerical schemes. One of the goals of this paper is to construct new closed form solutions complementing these results.

For equations that appear in L^∞ variational problems, while their analytic properties are thoroughly investigated by many authors, the construction and study of exact solutions is not thoroughly treated. There are many successful methods of constructing exact solutions for nonlinear PDEs. More often than not, these methods rely on some special algebro-geometric or analytic properties of the PDE, Darboux-Bäcklund transformations, inverse scattering transform, Painlevè property, etc. see [For90] for a review. These are connected to the integrable character of the equation. On the other hand, Lie group theory is general and makes little assumptions on the form of the PDE and hence it can be applied to strongly nonlinear and nonevolution equations such equation (1.1). There are many modern generalisations of Lie's classical approach. Examples of such generalisations are the *nonclassical symmetries* [BC69, CM94] and *approximate symmetries* [BGI89], to name a few. A detailed exposition of the classical theory can be found in the books [BA08, Hyd00, Olv93, Ovs82, Ste89] and the review papers [Oli10, Win93] and in references given therein. See also [Yag88] for a historic account and [DT06, DT04, Her97] for the implementation of these ideas using computer packages of symbolic algebra. Group theoretic methods in the study of differential equations have been applied successfully to problems arising from geometry, general theory of relativity, gas dynamics, hydrodynamics and many more, see [Ibr93].

In this work we restrict our attention to the case $n = 2$. We use x and y for the independent variables and so (1.1) simplifies to

$$(1.6) \qquad (f[u]_x)^2 u_{xx} + 2f[u]_x f[u]_y u_{xy} + (f[u]_y)^2 u_{yy} = 0$$

where now

$$(1.7) \qquad f[u] := (u_{xx})^2 + 2(u_{xy})^2 + (u_{yy})^2.$$

The aim of this paper is the construction of explicit ∞-Polyharmonic functions, i.e. solutions of equation (1.6). Towards this end we obtain the full Lie symmetry group for both equations (1.4) and (1.6) and we obtain one dimensional Lie subalgebras which we use to define appropriate canonical variables and reduce our PDEs to ordinary differential equations (ODEs). Studying the reduced ODEs we construct several new interesting invariant solutions. We also propose a conjecture for the symmetry structure of the general $\infty-$Polylaplacian in n dimensions. With this work we aim to promote group theoretic ideas in the study of these strongly nonlinear problems and their exact solutions, i.e. the $\infty-$Polyharmonic functions [KP18] and find potential minimizers for problems arising in the L^∞ variational calculus.

The paper is organised as follows: In the following section we study the reduced ∞-Polylaplacian and consider some of its algebraic properties. In Section 3 we briefly introduce some basics of Lie symmetries of differential equations and we fix the notation. Moreover, we derive the infinitesimal invariance conditions for both the $\infty-$Polylaplacian and its reduced version. We solve the determining equations in Section 4 and thus obtain the Lie algebras of the full symmetry groups of both equations. We also present some of the algebraic properties of the Lie algebras and we obtain the corresponding Lie symmetries. We also present a conjecture about the Lie symmetries of equations (1.1) and (1.4) for arbitrary n. In Section 5 we present a

partial classification of inequivalent generators under the action of the adjoint representation, i.e. the action of the symmetry group to its Lie algebra for both equations. We perform the corresponding reductions to ODEs and we construct new invariant solutions. We conclude in Section 6 with a summary and a discussion of our results.

2. The reduced $\infty-$Polylaplacian equation

In this section we introduce the reduced $\infty-$Polylaplacian equation over $\mathbb{R}$, describe some of its properties, and discuss some possible extensions of this work over algebraically closed fields of characteristic zero.

From the form of equation (1.6) it follows that the equation $f[u] = c$, where c is constant, defines a submanifold in the space of solutions of equation (1.6). One can observe that the real valued solutions of $f[u] = c$ which are not affine polynomials in x and y can always be rescaled to real solutions of the following equation

$$(2.1) \qquad f[u] = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 = 1$$

and vice versa. We call equation (2.1) the reduced ∞-Polylaplacian. Alternatively, one can study the equation $f[u] = c$ for any constant $c \in \mathbb{C}$ over $\mathbb{C}$-valued functions. For example, let $\mathcal{A} = \mathbb{C}[x, y]$ be the ring of polynomials in variables x and y with coefficients in $\mathbb{C}$ and with the usual gradation

$$\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k, \quad \mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$$

where $\mathcal{A}_k$ is the homogeneous component of all polynomials in $\mathcal{A}$ of degree k. Then, for simplicity, we can search for solutions of $f[u] = c$ in each subspace $\mathcal{A}_k$. We observe that $\mathcal{A}_0 \oplus \mathcal{A}_1 \subset \ker f$ and that $f : \mathcal{A}_k \to \mathcal{A}_{2k-4}$ for all $k \geq 2$. Specifically, for $k = 2$ we have that $f[\mathcal{A}_2] = \mathcal{A}_0 = \mathbb{C}$ and thus it follows that for any $c \in \mathbb{C}$ the equation $f[u] = c$ admits the solution $u = \alpha x^2 + \beta xy + \gamma y^2$ if and only if the parameters $(\alpha, \beta, \gamma) \in \mathbb{C}^3$ are elements of the $1-$parametric family of affine varieties $V(4\alpha^2 + 2\beta^2 + 4\gamma^2 - c)$. For $k > 2$ and since $f[\mathcal{A}_k] \subset \mathcal{A}_{2k-4}$ it follows that necessarily $c = 0$ and thus we only have to consider the equation $f[u] = 0$. Moreover, the parameter space associated to a solution $u \in \mathcal{A}_k$ has dimension $\dim \mathcal{A}_k = k + 1$ while the image has dimension $\dim \mathcal{A}_{2k-4} = 2k - 3$. It follows that f maps

$$u = \sum_{i+j=k} \alpha_{i,j} x^i y^j \mapsto f[u] = \sum_{m+n=2k-4} F_{m,n}(\alpha) x^m y^n$$

where $F_{m,n}$ are homogeneous quadratic polynomials of $\alpha_{i,j}$ and thus u will satisfy equation $f[u] = 0$ if and only if $\alpha = (\alpha_{k,0}, \alpha_{k-1,1}, \ldots, \alpha_{0,k}) \in \mathbb{C}^{k+1}$ is an element of the variety $V(I) \subset \mathbb{C}^{k+1}$, where I is the ideal generated by all $F_{m,n}$. Effectively, to find solutions of $f[u] = 0$ in $\mathcal{A}_k$ one has to solve $2k - 3$ quadratic equations for $k + 1$ variables (the parameter space) and thus for $k \geq 5$ the system is overdetermined. In principle these equations can be investigated using Gröbner basis, see [CLO92] and references therein, or numerical schemes. To investigate the existence and the form of solutions in $\mathcal{A}_k$ for all k is an open problem. More generally, the problem of characterising and classifying the solutions of $f[u] = c$ over the ring $\mathbb{F}[x, y]$, where $\mathbb{F}$ is an algebraically closed field of characteristic zero, is of particular mathematical interest and under current investigation by the authors. In this paper we will not pursuit these ideas any further, instead we assume that $u(x, y) \in \mathbb{R}$ and focus only on the Lie symmetries of the equation (2.1).

3. Infinitesimal invariance and determining equations

In this section we derive the determining equations of the generators of the symmetry group for both the ∞-Polylaplacian (1.6) and it's reduction (2.1).

A Lie point symmetry of equation (1.6) is a flow

$$(3.1) \qquad (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (e^{\epsilon X} x, e^{\epsilon X} y, e^{\epsilon X} u)$$

generated by a vector field

$$(3.2) \qquad X = \xi_1(x, t, u)\frac{\partial}{\partial x} + \xi_2(x, t, u)\frac{\partial}{\partial y} + \eta(x, t, u)\frac{\partial}{\partial u},$$

such that $\widetilde{u}(\widetilde{x}, \widetilde{y})$ is a solution of (1.6) whenever $u(x, y)$ is a solution of (1.6). As usual, we denote by $e^{\epsilon X}$ the *Lie series* $\sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} X^k$ with $X^k = XX^{k-1}$ and $X^0 = 1$.

To find the symmetries of equation (1.6) (resp. equation (2.1)) we have to solve the infinitesimal invariance condition for the vector field (3.2). In the case of equation (1.6) (resp. (2.1)) we have to use the third prolongation of X [BA08, Olv93, Ste89], namely $X^{(3)}$ (resp. the second prolonged vector field $X^{(2)}$). The calculations become extremely cumbersome and this is why we use a Mathematica based algebraic package called SYM [DT06, DT04] in order to obtain the infinitesimal symmetry conditions. The original infinitesimal symmetry condition for equation (1.6) reads

$$(3.3) \qquad\qquad X^{(3)} \Pi_{\infty}^2 u = 0 \mod (\Pi_{\infty}^2 u = 0)$$

and decomposes to a large overdetermined system of linear PDEs for ξ_1, ξ_2 and η known as *determining equations*. Using computer algebra and algorithms from computational differential algebra [Rei90, Sch07], we can prove that the infinitesimal invariance condition (3.3) is equivalent to the following system of 16 equations:

$$(3.4) \qquad\qquad \xi_{iu} = \xi_{ixx} = \xi_{ixy} = \xi_{iyy} = 0, \quad i = 1, 2$$

$$(3.5) \qquad\qquad \eta_{xx} = \eta_{xy} = \eta_{yy} = \eta_{xu} = \eta_{yu} = \eta_{uu} = 0,$$

$$(3.6) \qquad\qquad \xi_{1y} + \xi_{2x} = \xi_{1x} - \xi_{2y} = 0.$$

Similarly, the infinitesimal invariance condition for equation (2.1)

$$(3.7) \qquad\qquad X^{(2)} \left(f[u] - 1 \right) = 0 \mod (f[u] = 1)$$

is equivalent to the following system of 12 equations:

$$(3.8) \qquad\qquad \xi_{2xx} = \xi_{2u} = \xi_{1y} + \xi_{2x} = \xi_{1x} - \xi_{2y} = 0,$$

$$(3.9) \qquad\qquad \eta_{xx} = \eta_{xy} = \eta_{yy} = \eta_u - 2\xi_{1x} = 0,$$

$$(3.10) \qquad\qquad \xi_{1xx} = \xi_{1xy} = \xi_{1xu} = \xi_{1u} = 0.$$

Solutions of the overdetermined system of linear PDEs (3.4)-(3.6) (resp. (3.8)-(3.10)) will result to the algebra of the symmetry generators (3.2) of equation (1.6) (reps. (2.1)).

4. Lie symmetries of the ∞-Polylaplacian

In this section we focus our attention to the systems of equations (3.4)-(3.6) and (3.8)-(3.10). These equations form an overdetermined system of linear partial differential equations and thus it is possible that they only admit the trivial solution $\xi_1 = \xi_2 = \eta = 0$. This implies that the only Lie symmetry of equation (1.6) is the identity transformation. In what follows we will see that this is not the case. In this way we obtain the Lie algebra for the symmetry generators for both equations (1.6), (2.1) and thus, using the Lie series, derive the full groups of Lie point symmetries for both equations. At the end of this section we discuss about the discrete symmetries of the equations (1.6), and (2.1).

The general solution of the determining equations (3.4)-(3.6) is given by

$$(4.1) \qquad \xi_1 = c_1 x + c_2 y + c_3, \quad \xi_2 = -c_2 x + c_1 y + c_4, \quad \eta = c_5 x + c_6 y + c_7 u + c_8,$$

where c_i, $i = 1, \dots, 8$ are arbitrary real constants. It follows that the solution (4.1) defines an eight dimensional Lie algebra of generators where the obvious basis is formed by the following vector fields

$$(4.2) \qquad\qquad X_1 = \frac{\partial}{\partial x}, \ X_2 = \frac{\partial}{\partial y}, \ X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$(4.3) \qquad\qquad X_5 = u\frac{\partial}{\partial u}, \ X_6 = x\frac{\partial}{\partial u}, \ X_7 = y\frac{\partial}{\partial u}, \ X_8 = \frac{\partial}{\partial u}.$$

Similarly, for equation (2.1), we have that the general solution of the determining equations (3.8)-(3.10) is given by

$$(4.4) \qquad \xi_1 = c_1 x - c_2 y + c_3, \quad \xi_2 = c_2 x + c_1 y + c_4, \quad \eta = c_5 x + c_6 y + 2c_1 u + c_7,$$

where c_i, $i = 1, \dots, 7$ are arbitrary real constants. The Lie algebra of vector fields defined by the solution (4.4) is similar to the algebra spanned by the vector fields (4.2)-(4.3). It is a seven dimensional Lie algebra

spanned by the vector fields

$$Y_1 = \frac{\partial}{\partial x}, \ Y_2 = \frac{\partial}{\partial y}, \ Y_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, Y_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}, \tag{4.5}$$

$$Y_5 = x\frac{\partial}{\partial u}, \ Y_6 = y\frac{\partial}{\partial u}, \ Y_7 = \frac{\partial}{\partial u}. \tag{4.6}$$

The reason for this symmetry breaking is because the reduced equation (2.1) is not homogeneous and hence the equation only admits the scaling that makes each individual term, u_{xx}, u_{xy} and u_{yy} invariant, i.e. the symmetry generated by Y_4.

We denote the eight dimensional real Lie algebra by $\mathfrak{g}$ and the seven dimensional real Lie algebra by $\mathfrak{h}$, viz.

$$\mathfrak{g} = \mathrm{Span}\{X_i, \ i = 1, \ldots, 8\}, \quad \mathfrak{h} = \mathrm{Span}\{Y_i, \ i = 1, \ldots, 7\}. \tag{4.7}$$

Then it follows that equation (1.6) admits the symmetry group generated by $\mathfrak{g}$ while the symmetries of equation (2.1) are generated by $\mathfrak{h}$. Moreover, since for both equations $\xi_{1u} = \xi_{2u} = 0$, it follows that the symmetry transformations of both (1.6) and (2.1) are *fibre preserving transformations*.

Both Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are *solvable*. Indeed, we have that for both algebras the *derived series*

$$\mathfrak{g}^{(n)} = \left[\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}\right], \quad \mathfrak{g}^{(0)} = \mathfrak{g}$$

terminate to the trivial Lie algebra $\mathfrak{o} = \{0\}$ for a positive integer n. As usual $[\cdot, \cdot]$ denotes the commutator of vector fields which is the *Lie bracket* of $\mathfrak{g}$ and $\mathfrak{h}$. The first derived algebra, which is an ideal of $\mathfrak{g}$, is

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \mathrm{Span}\{X_1, X_2, X_6, X_7, X_8\}$$

and

$$\mathfrak{h}^{(1)} = \mathrm{Span}\{Y_1, Y_2, Y_5, Y_6, Y_7\}$$

as can be verified by inspecting Table 1 and Table 2.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	X_2	X_1	0	X_8	0	0
X_2	0	0	$-X_1$	X_2	0	0	X_8	0
X_3	$-X_2$	X_1	0	0	0	$-X_7$	X_6	0
X_4	$-X_1$	$-X_2$	0	0	0	X_6	X_7	0
X_5	0	0	0	0	0	$-X_6$	$-X_7$	$-X_8$
X_6	$-X_8$	0	X_7	$-X_6$	X_6	0	0	0
X_7	0	$-X_8$	$-X_6$	$-X_7$	X_7	0	0	0
X_8	0	0	0	0	X_8	0	0	0

TABLE 1.   Commutation relations of the Lie algebra $\mathfrak{g}$.

Similarly, we have that

$$\mathfrak{g}^{(2)} = \mathrm{Span}\{X_8\}, \quad \mathfrak{h}^{(2)} = \mathrm{Span}\{Y_7\}$$

and thus $\mathfrak{g}^{(3)} = \mathfrak{h}^{(3)} = \mathfrak{o}$.

$[Y_i, Y_j]$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
Y_1	0	0	Y_2	Y_1	Y_7	0	0
Y_2	0	0	$-Y_1$	Y_2	0	Y_7	0
Y_3	$-Y_2$	Y_1	0	0	$-Y_6$	Y_5	0
Y_4	$-Y_1$	$-Y_2$	0	0	$-Y_5$	$-Y_6$	$-2Y_7$
Y_5	$-Y_7$	0	Y_6	Y_5	0	0	0
Y_6	0	$-Y_7$	$-Y_5$	Y_6	0	0	0
Y_7	0	0	0	$2Y_7$	0	0	0

TABLE 2.   Commutation relations of the Lie algebra $\mathfrak{h}$.

Since the Lie algebra $\mathfrak{g}$ is solvable it admits a unique maximal ideal which is nilpotent and is called *nilradical*. It can be verified that $\mathfrak{g}$ can be written as the semi-direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus_s \mathfrak{g}_2$, where $\mathfrak{g}_1 = $

Span$(X_1, X_2, X_6, X_7, X_8)$ is its nilradical, $\mathfrak{g}_2 = $ Span(X_3, X_4, X_5) is the three dimensional abelian subalgebra, and thus the following relations hold

(4.8) $$[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_1, \quad [\mathfrak{g}_2, \mathfrak{g}_2] = 0, \quad [\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_1.$$

The nilradical $\mathfrak{g}_1$ has appeared in a classification of five dimensional nilpotent Lie algebras. Indeed, $\mathfrak{g}_1$ is isomorphic to the Lie algebra $\mathfrak{n}_{5,3}$, see page 231 in [ŠW14]. Similarly, the Lie algebra $\mathfrak{h}$ can be decomposed as a semi-direct sum of its nilradical $\mathfrak{h}_1 = $ Span$(Y_1, Y_2, Y_5, Y_6, Y_7)$ and of the abelian subalgebra $\mathfrak{h}_2 = $ Span(Y_3, Y_4).

It is important to mention that the Lie subalgebra Span$\{X_1, X_2, X_3\}$ is the Euclidean Lie algebra $e(2)$ formed by the *Killing vector fields* of $\mathbb{R}^2$. Thus equation (1.6) inherits the symmetries of the metric structure of $\mathbb{R}^2$ as it was also pointed out for the Aronsson's equation (1.5) in [FF11]. The extension $\widetilde{\mathfrak{g}} = $ Span$\{X_1, X_2, X_3, X_4, X_5\}$ of the Lie algebra $e(2)$ is still a Lie subalgebra of $\mathfrak{g}$. Finally, as can be seen by the Table 1, the Lie subalgebra Span$\{X_6, X_7, X_8\}$ forms an abelian ideal of $\mathfrak{g}$. Similar things can be said for the Lie algebra $\mathfrak{h}$. Here $\widetilde{\mathfrak{h}} = $ Span$\{Y_1, Y_2, Y_3, Y_4\}$ and Span$\{Y_5, Y_6, Y_7\}$ is also abelian ideal. We will use the subalgebras $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{h}}$ in the next section in the construction of invariant solutions. The identification of the structural properties of the symmetry Lie algebras is important since they can be used for deciding whether or not another PDE can be mapped to the equation at hand. Additionally, depending on the structural properties of the Lie algebra (simple, semi-simple, etc.) there are methods for the complete classification of its subalgebras. For example, it is possible to use the decomposition of $\mathfrak{g}$ (4.8) in order to fully classify all its one-dimensional subalgebras into conjugation classes using algorithms presented in [PWZ75], see also [Win90] for a review.

Using the Lie series we find that the full group of Lie point symmetries of (1.6) is generated by:

$$G_1 : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (x + \epsilon, y, u),$$
$$G_2 : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (x, y + \epsilon, u),$$
$$G_3 : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (x \cos\epsilon - y \sin\epsilon, x \sin\epsilon + y \cos\epsilon, u),$$
$$G_4 : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (e^\epsilon x, e^\epsilon y, u),$$
$$G_5 : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (x, y, e^\epsilon u),$$
$$G_6 : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (x, y, u + \epsilon x),$$
$$G_7 : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (x, y, u + \epsilon y),$$
$$G_8 : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (x, y, u + \epsilon).$$

Using the Lie symmetries G_i we can construct new solutions from known solutions. It follows that if $u = g(x, y)$ is a solution of equation (1.6) then the following:

$$\begin{array}{ll} u = g(x, y) + \epsilon x, & u = g(x - \epsilon, y), \\ u = g(x, y) + \epsilon y, & u = g(x, y - \epsilon), \\ u = g(x, y) + \epsilon, & u = g(x \cos\epsilon + y \sin\epsilon, -x \sin\epsilon + y \cos\epsilon), \\ u = e^\epsilon g(x, y), & u = g(e^{-\epsilon} x, e^{-\epsilon} y) \end{array}$$

are also solutions for all $\epsilon \in \mathbb{R}$. We obtain a similar symmetry structure for the reduced $\infty-$Polylaplacian equation (2.1) with the only difference being a breaking of the scaling symmetries, resulting in one fewer. More precisely, $G_1 - G_3$ and $G_6 - G_8$ are also Lie symmetries of (2.1) but G_4 and G_5 are not. Instead, equation (2.1) admits the following scaling symmetry

$$H : (x, y, u) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{u}) = (e^\epsilon x, e^\epsilon y, e^{2\epsilon} u)$$

which implies that if $u = g(x, y)$ is a solution of (2.1) then so is

$$u = e^{2\epsilon} g(e^{-\epsilon} x, e^{-\epsilon} y),$$

for all $\epsilon \in \mathbb{R}$. The scaling H can be seen as the composition of the scaling transformations G_4 and G_5 making each differential monomial on the left hand side of equation (2.1) invariant under the action of H.

Equations (1.6) and (2.1) also admit discrete symmetries. In particular, both equations (1.6) and (2.1) are invariant under the permutation σ of the independent variables x and y, as well as the x-reflection $\rho : x \mapsto -x$. Obviously, the y-reflection is also a symmetry of both equations and can be expressed as

$\sigma \circ \rho \circ \sigma$. These are also symmetries of the general ∞-Polylaplacian (1.1) in any dimension n. Moreover, (2.1), while not scaling invariant in the u-direction, remains invariant under the reflection $u \mapsto -u$.

The study of the Lie symmetry structure of the ∞-Polylaplacian for dimensions $n \geq 3$ and its corresponding reduction (2.1) is an open problem which is left for future work. Nevertheless, we formulate the following:

Conjecture: The Lie algebra of the symmetry generators of $\infty-$Polylaplacian equation (1.1) in n independent variables has dimension $3 + n(n+3)/2$ and it is spanned by the vector fields related to translations in the independent variables, scalings and affine linear translations in the dependent variable

$$
\text{(4.9)} \qquad \frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial u}, \quad \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}, \quad x_i \frac{\partial}{\partial u}, \quad i = 1, \ldots, n,
$$

as well as rotation symmetries generated by

$$
\text{(4.10)} \qquad -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}, \ 1 \leq i < j \leq n.
$$

Similarly, the symmetry algebra of the reduced $\infty-$Polylaplacian equation (1.4) in n independent variables has dimension $2 + n(n+3)/2$ and is spanned by the same generators for translation, rotations and affine linear translations in the dependent variable but with a scaling symmetry generated by

$$
\text{(4.11)} \qquad \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} + 2u \frac{\partial}{\partial u}.
$$

The above conjecture has been verified by the authors for $n = 3$.

5. Symmetry reductions and invariant solutions

In this section we are concerned with the symmetry reductions and the construction of group invariant solutions of equations (1.6) and (2.1). We construct solutions that are invariant under one dimensional subgroups acting non-trivially on the independent variables. More specifically, we focus on the symmetry subalgebras $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{h}}$ and we classify their one dimensional Lie subalgebras into equivalence classes under the action of the corresponding group. As already mentioned in section 4 by focussing on these subalgebras we will not obtain a full classification, however the problem is tractable and we are focussing on symmetries that have a physical meaning. In particular, some of the explicit solutions of the corresponding reduced ODEs are related to the results of numerical experimentation in [KP17]. The complete classification is left as a future work.

We first consider the reductions of equation (1.6) using one dimensional subalgebras of $\mathfrak{g}$ spanned by $X_i, \ i = 1, \ldots, 5$. To classify all the one dimensional subalgebras of $\widetilde{\mathfrak{g}} = \text{Span}\{X_1, \ldots, X_5\}$ we need to consider the action of the adjoint representation of the symmetry group of equation (1.6) on $\widetilde{\mathfrak{g}}$. The *adjoint representation* of a Lie group to its algebra is a group action and is defined by conjugacy as follows

$$
\text{Ad}_{\exp \epsilon X}(Y) = e^{\epsilon X} Y e^{-\epsilon X} = e^{\epsilon \text{ad}_X(Y)} = Y + \epsilon \text{ad}_X(Y) + \frac{\epsilon^2}{2!} \text{ad}_X^2(Y) + \cdots,
$$

where X and Y are elements of the Lie algebra and $\text{ad}_X(Y) = [X, Y]$, see for example [Olv93]. For the sake of completeness we present the adjoint representation of the symmetry group of (1.6) on its whole Lie algebra $\mathfrak{g}$ in Table 3 and of the symmetry group of (2.1) to $\mathfrak{h}$ in Table 4.

Ad	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	X_2	$X_3 + \epsilon X_2$	$X_4 + \epsilon X_1$	X_5	$X_6 + \epsilon X_8$	X_7	X_8
X_2	X_1	X_2	$X_3 - \epsilon X_1$	$X_4 + \epsilon X_2$	X_5	X_6	$X_7 + \epsilon X_8$	X_8
X_3	$\cos\epsilon X_1$ $-\sin\epsilon X_2$	$\sin\epsilon X_1$ $+\cos\epsilon X_2$	X_3	X_4	X_5	$\cos\epsilon X_6$ $-\sin\epsilon X_7$	$\sin\epsilon X_6$ $+\cos\epsilon X_7$	X_8
X_4	$e^{-\epsilon}X_1$	$e^{-\epsilon}X_2$	X_3	X_4	X_5	$e^{\epsilon}X_6$	$e^{\epsilon}X_7$	X_8
X_5	X_1	X_2	X_3	X_4	X_5	$e^{-\epsilon}X_6$	$e^{-\epsilon}X_7$	$e^{-\epsilon}X_8$
X_6	$X_1 - \epsilon X_8$	X_2	$X_3 + \epsilon X_7$	$X_4 - \epsilon X_6$	$X_5 + \epsilon X_6$	X_6	X_7	X_8
X_7	X_1	$X_2 - \epsilon X_8$	$X_3 - \epsilon X_6$	$X_4 - \epsilon X_7$	$X_5 + \epsilon X_7$	X_6	X_7	X_8
X_8	X_1	X_2	X_3	X_4	$X_5 + \epsilon X_8$	X_6	X_7	X_8

TABLE 3. The $Ad_{\exp \epsilon X_i} X_j$ is shown in the (i, j) entry of the table.

Ad	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
Y_1	Y_1	Y_2	$Y_3 + \epsilon Y_2$	$Y_4 + \epsilon Y_1$	$Y_5 + \epsilon Y_7$	Y_6	Y_7
Y_2	Y_1	Y_2	$Y_3 - \epsilon Y_1$	$Y_4 + \epsilon Y_2$	Y_5	$Y_6 + \epsilon Y_7$	Y_7
Y_3	$\cos\epsilon Y_1$ $-\sin\epsilon Y_2$	$\sin\epsilon Y_1$ $+\cos\epsilon Y_2$	Y_3	Y_4	$\cos\epsilon Y_5$ $-\sin\epsilon Y_6$	$\sin\epsilon Y_5$ $+\cos\epsilon Y_6$	Y_7
Y_4	$e^{-\epsilon}Y_1$	$e^{-\epsilon}Y_2$	Y_3	Y_4	$e^{-\epsilon}Y_5$	$e^{-\epsilon}Y_6$	$e^{-2\epsilon}Y_7$
Y_5	$Y_1 - \epsilon Y_7$	Y_2	$Y_3 + \epsilon Y_6$	$Y_4 + \epsilon Y_5$	Y_5	Y_6	Y_7
Y_6	Y_1	$Y_2 - \epsilon Y_7$	$Y_3 - \epsilon Y_5$	$Y_4 + \epsilon Y_6$	Y_5	Y_6	Y_7
Y_7	Y_1	Y_2	Y_3	$Y_4 + 2\epsilon Y_7$	Y_5	Y_6	Y_7

TABLE 4. The $Ad_{\exp \epsilon Y_i} Y_j$ is shown in the (i,j) entry of the table.

Any one dimensional subalgebra of $\widetilde{\mathfrak{g}}$ is equivalent, under the adjoint representation, to one of the following cases:

$$
\begin{array}{llll}
(A1) & X_1, & (A5) & \alpha X_3 + X_5, \\
(A2) & X_3, & (A6) & \alpha X_4 + X_5, \\
(A3) & X_4, & (A7) & \gamma X_1 + \alpha X_3 + X_4, \\
(A4) & \gamma X_1 + X_5 & (A8) & \gamma X_1 + \alpha X_3 + \beta X_4 + X_5,
\end{array}
$$

where $\gamma \in \{0,1\}$ and $\alpha, \beta \in \mathbb{R}\backslash\{0\}$. Starting with a general element of $\widetilde{\mathfrak{g}}$ of the form

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5$$

we can use all $Ad_{\exp \epsilon X_i}$ in order to simplify as much as possible and effectively classify all different one dimensional subalgebras of $\widetilde{\mathfrak{g}}$. The adjoint action Ad induces an action on the coefficients α_i, i.e. on $\mathbb{R}^5$. We observe that α_3, α_4 and α_5 are invariants of the induced action. This implies that we can classify all inequivalent vector fields according to whether these invariants are zero or not. Moreover, we can rescale the vector field X, use the permutation symmetry σ and the reflection symmetries ρ and $\sigma \circ \rho \circ \sigma$ to identify some subcases and thus simplify our classification list. For example, in the case where $\alpha_3 = \alpha_4 = \alpha_5 = 0$ we act with $Ad_{\exp \epsilon X_3}$ to X and we obtain $\widetilde{X} = (\alpha_1 \cos(\epsilon) + \alpha_2 \sin(\epsilon))X_1 + (\alpha_2 \cos(\epsilon) - \alpha_1 \sin(\epsilon))X_2$. Choosing $\epsilon = \arctan(\alpha_2 \alpha_1^{-1})$ and multiplying by a constant factor we obtain X_1. The other cases are obtained in a similar manner but the calculations are omitted for simplicity. The interested reader can find more details on such constructions as well as simpler examples in [Hyd00, Olv93, Ovs82].

Similar considerations hold for the symmetry algebra of equation (2.1) $\widetilde{\mathfrak{h}} = \mathrm{Span}\{Y_1, \ldots, Y_4\}$. In this case any one dimensional subalgebra of $\widetilde{\mathfrak{h}}$ is equivalent to one of the following cases:

$$
\begin{array}{llll}
(B1) & Y_1, & (B3) & Y_4, \\
(B2) & Y_3, & (B4) & \gamma Y_1 + \alpha Y_3 + Y_4,
\end{array}
$$

where $\gamma \in \{0,1\}$ and $\alpha \in \mathbb{R}\backslash\{0\}$. To prove this we use similar arguments. Beginning with a general element of $\widetilde{\mathfrak{h}}$ of the form

$$Y = \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 Y_3 + \beta_4 Y_4$$

we classify all inequivalent cases. In this case the invariants of the induced action are β_3 and β_4.

5.1. **Invariant solutions via symmetry reductions.** We proceed by first considering the symmetry reductions to ODEs and then continue constructing new solutions, of equation (1.6), which are invariant under the symmetry transformations corresponding to the vector fields A1-A8. We do the same for equation (2.1). We first consider the reductions and solutions of equation (1.6).

1. Solutions of (1.6) which are invariant under the symmetry generated by X_1 are of the form $u = g(y)$. This implies that g is a solution of the trivial ODE $g''(y)^3 g'''(y)^2 = 0$ and thus it follows that $u = c_1 y^2 + c_2 y + c_3$ satisfies the $\infty-$Polylaplacian. Since equation (1.6) admits the permutation σ and also contains derivatives of at least second order it is easy to verify that the general quadratic polynomial in x and y

$$(5.1) \qquad u = \sum_{0 \le i+j \le 2} c_{ij} x^i y^j$$

is also a solution.

2. Rotationally invariant solutions are of the form $u = g(s)$ where $s = x^2 + y^2$. The reduced equation is the following ODE for $f(s)$

(5.2)
$$(2sg_{ss} + g_s)\left[s(2sg_{ss} + g_s)g_{sss} + (3sg_{ss} + 2g_s)g_{ss}\right]^2 = 0.$$

The factorisation of the reduced equation implies that

(5.3)
$$u = \sqrt{x^2 + y^2},$$

is a solution of equation (1.6), which we obtain by solving the linear equation

$$2sg_{ss} + g_s = 0$$

and then changing to the original $x, y-$variables. Note that this solution is also the most general rotationally invariant solution of Aronsson's equation (1.5) in two independent variables [FF11]. However, equation (1.6) may admit more solutions of this type that correspond to the equation defined by the second factor in (5.2), i.e. a third order nonlinear ODE.

3. The quantities u and $s = xy^{-1}$ are algebraic invariants of the Lie group generated by X_4. We assume that $u = g(s)$ and we obtain a reduced differential equation for $g(s)$ which, similarly to the previous case can be decomposed to a product of two factors. One of these factors is too complicated to include it, however, the other factor is simpler and defines the differential equation

$$(1 + s^2)g_{ss} + 2sg_s = 0$$

from which we can obtain the solution

(5.4)
$$u = \arctan\left(\frac{x}{y}\right),$$

of equation (1.6). Using the permutation symmetry of the independent variable it follows that $\arctan(y/x)$ is also a solution. It can be easily verified that any linear combination of these two solutions is again a solution.

4. In the case of the generator $\gamma X_1 + X_5$ we have two subcases depending on the value of γ. If $\gamma = 0$ the only invariant solution is the trivial solution $u = 0$. If $\gamma = 1$ we have two invariants of the corresponding Lie group, namely $e^{-x}u$ and y. This implies that the most general form of an invariant solution is $u = e^x h(y)$. Substituting the ansatz for u in (1.6) we obtain the following equation for $h(y)$

$$4hg^2 + 4h_y g g_y + h_{yy}(g_y)^2 = 0, \quad g[h] := h^2 + 2(h_y)^2 + (h_{yy})^2.$$

This equation is difficult to solve and it does not admit any obvious factorisations as in the previous cases. It is important to note at this point that $g[h] = 0$ defines a subset of solutions. Obviously, if $h(y)$ is a real function then the only such solution is $h(y) = 0$. However, over the complex numbers equation $g[h] = 0$ might be tractable and have nontrivial solutions.

5. In the case of the generator $\alpha X_3 + X_5$ the invariants are $s = x^2 + y^2$ and $r = \arctan\left(\frac{x}{y}\right) + \alpha \ln(u)$. The most general solution invariant under the symmetry generated by $\alpha X_3 + X_5$ is of the form

$$u = \exp\left(\frac{1}{\alpha}g(s) - \frac{1}{\alpha}\arctan\left(\frac{x}{y}\right)\right).$$

The resulting reduced ODE for $g(s)$ is too complicated to handle or even write down. For specific values of α it might be possible to simplify the expressions, due to cancellations or factorisations, and thus find explicit solutions.

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6. The quantities $s = xy^{-1}$ and $r = ux^{-\frac{1}{\alpha}}$ are invariants under the action of the Lie symmetry generated by $\alpha X_3 + X_5$. In the limit $\alpha \to \infty$ we reduce to the generator X_3. For $\alpha \neq 0$, the most general invariant solution of (1.6) is of the form $u = x^{\frac{1}{\alpha}} g(s)$. For a general $\alpha \neq 0$ the reduced ODE it is complicated and we will not present it here. However, it is interesting to note that in the special case $\alpha = 1/2$ the reduced equation factorises as follows

$$E_1[g]^3 E_2[g]^2 = 0$$

where

$$E_1[g] := s^2(1+s^2)^2 g_{ss} + 2s(1+s^2)(2+s^2)g_s + 2g$$

and

$$E_2[g] := sg_{sss} + 6sg_{ss} + 6g_s.$$

Equations $E_1[g] = 0$ and $E_2[g] = 0$ are both linear and can be solved exactly. Solving the first equation we obtain the following solutions for (1.6)

(5.5)
$$u = (x^2 + y^2)\left[c_1 \cos\left(\sqrt{2}\arctan\left(\frac{x}{y}\right)\right) + c_2 \sin\left(\sqrt{2}\arctan\left(\frac{x}{y}\right)\right)\right]$$

for $c_i \in \mathbb{R}$. The general solution of equation $E_2[g] = 0$ can also be find and implies the solution

$$u = c_1 x^2 + c_2 xy + c_3 y^2$$

with $c_i \in \mathbb{R}$. It would be very interesting to find a method or some criteria which will detect possible values of the parameter in which such factorisations occur. Perhaps such suitable necessary conditions for the parameter α can be obtained using a Painlevé type analysis, see Chapter 7 of [AC91] and [Con12] for reviews and detailed references. It is interesting to notice that while the scaling $(x, y, u) \mapsto (e^{\alpha\epsilon}x, e^{\alpha\epsilon}y, e^{\epsilon}u)$ is a symmetry for every α, each of the differential monomials of equation (1.6) has the same weight, i.e.,

$$f_x^2 u_{xx} \mapsto e^{(5-12\alpha)\epsilon} f_x^2 u_{xx}$$

and similarly for the other terms. This observation implies that for the special value $\alpha = 12/5$ all three terms of equation (1.6) are individually invariant under the scaling symmetry. This observation further implies that

$$\Pi_\infty^2 x^r \sim x^{5r-12}$$

and because of symmetry the same will hold for y^r. Putting all these together it can be verified that

$$\Pi_\infty^2 (ax^r + by^r) \sim a^5 x^{5r-12} + b^5 y^{5r-12}$$

from where we obtain, for $r = 12/5$, a scaling invariant solution, also known as *similarity solution*, of equation (1.6) if and only if (a, b) satisfies

$$a^5 + b^5 = 0.$$

The only real solution is given by $b = -a$ and in this way we recover the solution

(5.6)
$$u = x^{12/5} - y^{12/5},$$

which was first constructed in [KP18]. The same arguments are valid in the case of the general $\infty-$Polylaplacian in n independent variables. In this case we obtain that

(5.7)
$$u = c_1 x_1^{12/5} + \cdots + c_n x_n^{12/5}$$

is a solution if and only if $(c_1, ..., c_n)$ lies on the affine variety $V(c_1^5 + \cdots + c_n^5)$. This is an invariant solution under the scaling symmetry generated by the vector field

$$X = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n} + \frac{12}{5}\frac{\partial}{\partial u}.$$

Indeed, it can be written in the following form

$$I_0 = c_1 I_1^{12/5} + \cdots + c_{n-1} I_{n-1}^{12/5} + c_n$$

where

$$I_0 = ux_n^{-12/5}, \quad I_j = x_j x_n^{-1}, \ j = 1, \cdots, n-1$$

are invariants of the scaling symmetry generated by X.

7. For the generator $\gamma X_1 + \alpha X_3 + X_4$ we have to consider the two subcases $\gamma = 0$ and $\gamma = 1$. When $\gamma = 0$ the generator is $\alpha X_3 + X_5$ with $\alpha \in \mathbb{R}\backslash\{0\}$ and we can verify that u and

$$s = \arctan\left(\frac{x}{y}\right) - \frac{\alpha}{2}\ln\left(x^2 + y^2\right)$$

are invariants under the action of the corresponding Lie group. The ansatz $u = g(s)$ leads to an ODE that is too complicated to include it here. Nevertheless, the reduced ODE admits a factorisation where one of the factors is given by

$$E_1[g] := (1 + \alpha^2)g_{ss} + \alpha g_s.$$

The linear ODE $E_1[g] = 0$ can be integrated for every α and gives the following solution of the ∞-Polylaplacian

(5.8)
$$u = \frac{(x^2 + y^2)^{\frac{\alpha^2}{2(1+\alpha^2)}}}{\exp\left(\frac{\alpha}{1+\alpha^2}\arctan\left(\frac{x}{y}\right)\right)}$$

Similarly, when $\gamma = 1$ the invariants are u and

$$z = -4\arctan\left(\frac{\alpha x + y}{1 + x - \alpha y}\right) + 2\alpha\ln\left[\alpha^2 + 2\alpha^2 x - 2\alpha^3 y + \alpha^2(1 + \alpha^2)(x^2 + y^2)\right]$$

and the reduced ODE for $h(z)$ contains the following linear factor

$$E_2[h] := 4(1 + \alpha^2)h_{zz} - \alpha h_z.$$

Solving the linear ODE $E_2[h] = 0$ for all α we finally obtain the following solution

(5.9)
$$u = \frac{\left[\alpha^2 + 2\alpha^2 x - 2\alpha^3 y + \alpha^2(1 + \alpha^2)(x^2 + y^2)\right]^{\frac{\alpha^2}{2(1+\alpha^2)}}}{\exp\left(\frac{\alpha}{1+\alpha^2}\arctan\left(\frac{\alpha x + y}{1 + x - \alpha y}\right)\right)}$$

of the ∞-Polylaplacian. It is interesting to notice that solution (5.8) can be seen as the dominant part of solution (5.9) as $\alpha \to \infty$.

Unfortunately the reductions that correspond to the generator $A8$ are too complicated to handle. We now focus on the reductions of the reduced $\infty-$Polylaplacian (2.1) that give additional information. Solutions of equation (2.1) that are invariant under translation in the x-direction are of the form $u = g(y)$ where f satisfies

$$0 = g_{yy}^2 - 1 = (g_{yy} + 1)(g_{yy} - 1).$$

The solutions of these equations are just quadratic polynomials in y and thus add nothing new. Solutions invariant under rotations are of the form $u = g(s)$ where $s = x^2 + y^2$ and $g(s)$ satisfies the following ODE

$$16s^2 g_{ss}^2 + 16sg_s g_{ss} + 8g_s^2 - 1 = 0.$$

The general solution of this ODE is not known, nevertheless a simple polynomial ansatz can lead to the special solution $s/2\sqrt{2}$. The corresponding solution of the (2.1) and hence of (1.6) is contained in the family of polynomial solutions. Finally, the ansatz $u = x^2 g(s)$ where $s = xy^{-1}$ leads to solutions that are invariant under the Lie symmetry generated by Y_4. In this case the reduced ODE for g is given by

$$s^4(1 + s^2)^2 g_{ss}^2 + 4s^2(g + s(2 + 3s^2 + s^4)g_s)g_{ss} + 2s^2(8 + 9s^2 + 2s^4)g_s^2 + 16sgg_s + 4g^2 = 1.$$

As before a Laurent polynomial ansatz leads to the special solutions $(\sqrt{2}s)^{-1}$ and $(2s^2)^{-1}$. The corresponding solutions of the $\infty-$Polylaplacian are contained in the polynomial family. Due to the complexity of the expressions we didn't manage to obtain something meaningful in the final case $(B4)$.

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6. Conclusions and discussion

In this paper we studied the ∞-Polylaplacian equation (1.6) and the reduced ∞-Polylaplacian equation (2.1) in two dimensions ($n = 2$) from an algebraic point of view. The latter can be seen as a second order analogue of the Eikonal equation. For both equations we found the complete group of Lie point symmetries and we classified all the non-equivalent, under the adjoint action, one dimensional Lie subalgebras of $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{h}}$ that correspond to translations, rotations and scalings. For each generator in our list we constructed canonical invariant coordinates and used them to perform the corresponding symmetry reduction. We studied the obtained reduced ODEs and constructed many new self-similar special solutions (5.1), (5.3), (5.4), (5.5), (5.6), (5.8), (5.9). The family of functions (5.7), parametrised by an affine variety, is a solution of (1.1) for every n. The complete classification of one dimensional subalgebras of the full symmetry algebras $\mathfrak{g}$ and $\mathfrak{h}$ together with a more in depth analysis of all of their invariant solutions is still an open problem and is left for future work. It is also interesting to investigate the structure of the solutions for $n \geq 3$. As a first step towards this direction we presented a conjecture on the full group of Lie point symmetries of the ∞-Polylaplacian and its reduced version in n-dimensions.

We believe that for this type of strongly nonlinear PDE that arise in calculus of variations in L^∞ deep intuition can be gained by studying the structure of their Lie symmetries. There are also many related open problems. For example, currently, to the best of the author's knowledge, a Noether-like theorem for these problems is not known. A topic of ongoing work is to investigate whether Noether's classical theorem applied in variational problems in L^p survives the limit $p \to \infty$.

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