A multiscale asymptotic theory of extratropical wave–mean flow interaction

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ABSTRACT

Multiscale asymptotic methods are used to derive wave-activity equations for planetary and synoptic scale eddies and their interactions with a zonal mean flow. The eddies are assumed to be of small amplitude, and the synoptic-scale zonal and meridional length scales are taken to be equal. Under these assumptions, the zonal-mean and planetary-scale dynamics are planetary geostrophic (i.e. dominated by vortex stretching), and the interaction between planetary and synoptic scale eddies occurs only through the zonal mean flow or through diabatic processes. Planetary scale heat fluxes are shown to enter the angular momentum budget through meridional mass redistribution. After averaging over synoptic length and time scales, momentum fluxes disappear from the synoptic-scale wave-activity equation whilst synoptic-scale heat fluxes disappear from the baroclinicity equation, leaving planetary-scale heat fluxes as the only adiabatic term coupling the baroclinic and barotropic components of the zonal mean flow. In the special case of weak planetary waves, the decoupling between the baroclinic and barotropic parts of the flow is complete with momentum fluxes driving the barotropic zonal mean flow, heat fluxes driving the wave activity, and diabatic processes driving baroclinicity. These results help explain the apparent decoupling between the baroclinic and barotropic components of flow variability recently identified in observations, and may provide a means of better understanding the link between thermodynamic and dynamic aspects of climate variability and change.

1. Introduction

The interaction between jet variability and eddies is a long-studied topic, but the interaction is not yet understood well enough to identify causal mechanisms for variability or sources of systematic errors in models. There are well-developed theoretical frameworks for the zonally homogeneous case (e.g. annular-mode variability), however zonally asymmetric analyses including planetary scale interactions are more complicated and only partial theories for this case exist (Hoskins et al. 1983; Plumb 1985, 1986). Yet longitudinal variations and synoptic-planetary scale interactions are important for the location and strength of the storm tracks and blocking episodes (Hoskins et al. 1983; Luo 2005; Simpson et al. 2014). These phenomena strongly affect the regional climate and its climate change. As the dynamical aspects of climate are not yet well understood, there is low confidence in circulation patterns simulated by global and regional models and their response to climate change (Shepherd 2014).

An important aspect of wave-mean flow interaction concerns barotropic and baroclinic processes and their links through eddy momentum and heat fluxes. It has recently been shown from observations for the Southern and Northern Annular Modes in Thompson and Woodworth (2014) and Thompson and Li (2015) that the zonal mean flow is affected only by momentum fluxes and not by heat fluxes, while the opposite is true for a so-called baroclinic annular mode (BAM) that is based on eddy kinetic energy (EKE). This decoupling goes against the usual Transformed Eulerian Mean (TEM) perspective, first introduced by Andrews and McIntyre (1976), within which both heat and momentum fluxes affect the zonal mean flow tendency through the Eliassen-Palm (EP) flux divergence. The decoupling was further investigated in Thompson and Barnes (2014), who found an oscillating relationship between EKE and heat flux with time periods of 20-30 days. A similar relationship was found between wave activity and heat flux in Wang and Nakamura (2015, 2016).

To derive a theoretical framework for understanding planetary-synoptic scale interactions and the apparent decoupling of the baroclinic and barotropic parts of the flow, we use multi-scale asymptotic methods as introduced in Dolaptchiev and Klein (2009, 2013) (hereafter DK09 and DK13, respectively). This approach is taken as such methods provide a self-consistent (albeit idealised) framework for studying interactions between processes on different length and time scales, starting from a minimal set of assumptions. While the derived theory using these methods may not be quantitatively accurate for the atmosphere, it can still provide qualitative value, especially when trying to determine the causal relationships that are so elusive in standard budget calculations. This is analogous to the use of the quasi-geostrophic approximation, which provides a clear qualitative picture of the large scale flow and both planetary and synoptic scale eddies, however for accurate representation of the flow (e.g. in weather prediction), the primitive equations are used. Therefore, the aim of this work is to find a theoretical framework by which to better understand the emergent properties of observations and model behavior, rather than developing a predictive theory.

DK13 used a separation of length scales in the meridional and zonal directions, with an isotropic scaling for the synoptic scales, as well as a temporal scale separation between the synoptic and planetary waves. Isotropic scaling for the synoptic scales is standard in quasi-geostrophic (QG) theory (Pedlosky 1987), and a meridional scale separation has been argued to be a useful and physically realizable idealization of baroclinic instability (Haidvogel and Held 1980). These assumptions allowed DK13 to study planetary and synoptic scale interactions. However, they did not derive a wave activity equation or develop explicit equations for the interaction with a zonal mean flow. These aspects are the focus of this paper. For simplicity, we derive the asymptotic equations for the case of small-amplitude eddies evolving in the presence of a zonal mean flow, which is an important special case of the DK13 framework. As well as giving a theoretical description for the interaction of a zonal mean flow with planetary and synoptic scale waves, this setting also allows a study of the link between baroclinic and barotropic processes, and the relative importance of planetary and synoptic scale waves for these processes.

The outline of the paper is as follows. Section 2 gives the equations and assumptions used to derive the potential vorticity (section 3), wave activity and mean flow equations (section 4), and the angular momentum budget for the zonal mean flow (section 5). The momentum, continuity, thermodynamic and vorticity equations at different asymptotic orders, which are needed for the derivations, are given in Appendix A. Further details on the derivations of the mean flow and angular momentum equations, and the non-acceleration theorem, are given in Appendices B, C and D. The zonally homogeneous case with weak planetary scale waves is discussed in section 6, and conclusions are given in section 7.

2. The multiscale asymptotic model

a. Nondimensional compressible flow equations

The asymptotic system of equations is derived starting from the nondimensionalised compressible flow equations in spherical coordinates with a small parameter ε[1] (DK09). To obtain the nondimensional equations the DK09 and DK13 scaling parameters[2] are used, based on the assumption that the waves are not propagating faster than the speed of sound. In this process,

[1] ε is defined as $\left(a^*\Omega^2 g^{-1}\right)^{1/3}$ (global atmospheric aspect ratio), where Ω is Earth's rotation rate, a^* is Earth's radius and g the Earth's gravitational acceleration. ε is a constant within the range $1/8$ to $1/6$.

[2] Pressure $p_{ref} = 10^5$ Pa, air density $\rho_{ref} = 1.25$ kg m^{-3}, characteristic flow velocity $u_{ref} = 10$ m s^{-1}, scale height $h_{sc} = p_{ref}/g\rho_{ref} \approx 10$ km, gravitational acceleration $g \approx 10$ m s^{-2}, and time scale $t_{ref} = h_{sc}/u_{ref} \approx 20$ min.

the following nondimensional numbers appear (DK09): Rossby[3] ($Ro_{QG} = u_{ref}/2\Omega L_{QG}$ with $L_{QG} = \varepsilon^{-2}h_{sc}$), Mach ($M = u_{ref}/\sqrt{p_{ref}/\rho_{ref}}$), Froude ($Fr = u_{ref}/\sqrt{gh_{sc}}$) and the ratio of density and potential temperature scale heights $\sqrt{h_{sc}/H_\theta}$. These are related to the small parameter ε according to $\sqrt{M} \approx \sqrt{Fr} \approx Ro_{QG} \approx \sqrt{h_{sc}/H_\theta} \approx \varepsilon$ (DK09). This procedure yields the system (the full derivation is given in DK09):

$$\frac{Du}{Dt} - \varepsilon^3\left(\frac{uv\tan\phi}{R} - \frac{uw}{R}\right) + \varepsilon(w\cos\phi - v\sin\phi) = -\frac{\varepsilon^{-1}}{R\rho\cos\phi}\frac{\partial p}{\partial\lambda} + S_u \tag{1a}$$

$$\frac{Dv}{Dt} + \varepsilon^3\left(\frac{u^2\tan\phi}{R} + \frac{vw}{R}\right) + \varepsilon u\sin\phi = -\frac{\varepsilon^{-1}}{R\rho}\frac{\partial p}{\partial\phi} + S_v \tag{1b}$$

$$\frac{Dw}{Dt} - \varepsilon^3\left(\frac{u^2}{R} + \frac{v^2}{R}\right) - \varepsilon u\cos\phi = -\frac{\varepsilon^{-4}}{\rho}\frac{\partial p}{\partial z} - \varepsilon^{-4} + S_w \tag{1c}$$

$$\frac{D\theta}{Dt} = S_\theta \tag{1d}$$

$$\frac{D\rho}{Dt} + \frac{\varepsilon^3\rho}{R\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\phi)}{\partial\phi}\right) + \rho\frac{\partial w}{\partial z} + \frac{\varepsilon^3 2w\rho}{R} = 0 \tag{1e}$$

$$\rho\theta = p^{1/\gamma} \tag{1f}$$

where S denotes source-sink terms ($S_{u,v,w}$ are the frictional terms, while S_θ represents diabatic effects), $\sin\phi = f$ is the nondimensional Coriolis parameter, p is nondimensional pressure, θ is nondimensional potential temperature, ρ is nondimensional density, (u, v, w) represent the nondimensional 3-D velocity field, $R = \varepsilon^3 r$, $r = \varepsilon^{-3}a + z$ where z is altitude from the ground, $a = a^*\varepsilon^3/h_{sc}$ is nondimensional Earth's radius, ϕ is latitude, λ is longitude, t is time, all parameters are nondimensional, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\varepsilon^3 u}{R\cos\phi}\frac{\partial}{\partial\lambda} + \frac{\varepsilon^3 v}{R}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}. \tag{2}$$

Note that the shallow-atmosphere limit $R \to a$ is used here unless otherwise stated (this approximation is used as it holds well to leading order). Expanding R, the material derivative (2) involves

[3]Note that the Rossby number (Ro) used in DK09 and DK13 is $\varepsilon^{-2}Ro_{QG}$ as they used the vertical instead of the horizontal length scale to define it.

horizontal advection terms $-a^{-1}\varepsilon^6 z(u\{a\cos\phi_p\}^{-1}\partial/\partial\lambda + va^{-1}\partial/\partial\phi)$ that become relevant at 5^{th} and higher orders.

b. Assumptions for multiscale asymptotic methods

In order to derive the multiscale asymptotic version of the equations, some assumptions must be made. In particular, we assume small-amplitude eddies in the presence of a zonal mean flow. This approximation is made in order to gain qualitative insight into the behavior of the system, and to allow connection with previous theories of wave, mean-flow interaction. This can be considered a special case of DK13, with the eddies (but not the zonal mean flow) scaled down by one order of ε. The assumptions for the scale separation between the synoptic, planetary and mean flow in time, height, latitude and longitude are given in Table 1 (following DK13), where the subscripts m, p and s represent mean, planetary and synoptic scales, respectively. Note that $\phi_s \gg \phi_p$ (similarly for other coordinates) since the same meridional distance is a much larger number when measured on synoptic scales compared to planetary or zonal mean scales. Here λ_m is not considered as the zonal mean flow is uniform in longitude, λ_p and ϕ_p represent variations of the flow on planetary scales (those of order a^*), λ_s and ϕ_s represent variations on synoptic scales (of order 1000 km), and the time scales are well separated between the mean flow, planetary and synoptic scale eddies, where t_s is of order a day, t_p is of order a week and t_m is a seasonal timescale. The time scales emerge naturally from the equations; t_m is ε^2 slower than t_p because the eddy fluxes driving the zonal mean flow changes are quadratic in eddy amplitude. (In the finite-amplitude theory of DK13, there is no distinction between the two timescales.) As this is the small-amplitude limit of the system, we expect that in practice the zonal mean flow time scale would be shorter. Note that from the above assumptions we see that there is a separation of scales in the meridional direction, which has implications for the final results (see further discussion in sections 3, 4 and 6).

Using these scales, we can write asymptotic series for all variables; an example for potential temperature (which provides stratification) is (following DK09, DK13):

$$\theta(\lambda,\phi,z,t) = 1+\varepsilon^2\theta^{(2)}(\phi_p,t_m,z)+\varepsilon^3\theta^{(3)}(\mathbf{X}_p,z)+\varepsilon^4\theta^{(4)}(\mathbf{X}_p,\mathbf{X}_s,z)+... \quad (3)$$

where the number in parentheses in superscript represents the order of the variable, $\mathbf{X}_p = (\lambda_p,\phi_p,t_p)$ and $\mathbf{X}_s = (\lambda_s,\phi_s,t_s)$. Here the first order term has been omitted as $h_{sc}/H_\theta \propto \Delta\theta/\theta_0 \approx \varepsilon^2$; to make this $\mathscr{O}(\varepsilon)$ would lead to stronger wind variations (of order 70 m s^{-1}) (DK09), which would require a different treatment. Note that here the leading order variation in potential temperature $\theta^{(2)}$ depends on ϕ_p and z, not only on z as is the case for the static stability parameter in QG theory.

In order to have a well defined asymptotic expansion (3) the sublinear growth condition (DK13) is required. This means that variables at any order grow slower than linearly in any of the synoptic coordinates, which effectively means that any averaging over the synoptic scales ($\mathbf{X}_s$) sets the derivatives over synoptic scales to zero (for more details see DK13).

The full set of equations at different asymptotic orders using the assumptions from this section is given in Appendix A. This includes the momentum, thermodynamic and continuity equations, thermal wind, hydrostatic balance and the vorticity equation. These equations are used in the following sections to derive potential vorticity, wave activity and mean flow equations.

3. Potential vorticity equation

To derive the potential vorticity (PV) equation, a vorticity equation has to be derived first. To do so (see Appendix A for the full derivation), take $\nabla_s \times \mathscr{O}(\varepsilon^3)$ momentum equation (A6) and use the $\mathscr{O}(\varepsilon^4)$ continuity equation (A15), which yields

$$\frac{\partial}{\partial t_s}\zeta^{(1)} + \mathbf{u}^{(0)}\cdot\nabla_s\zeta^{(1)} - \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\rho^{(0)}w^{(4)}\right) + \beta v^{(1)} = S_\zeta \quad (4)$$

where $\nabla_s = \left((a\cos\phi_p)^{-1}\partial/\partial\lambda_s, a^{-1}\partial/\partial\phi_s\right)$, $\mathbf{u}^{(0)} = u^{(0)}\mathbf{e}_\lambda$ is horizontal velocity of the mean flow, $\beta = a^{-1}\partial f/\partial\phi_p$, $\boldsymbol{\zeta}^{(1)} = \zeta^{(1)}\mathbf{e}_r = \nabla_s \times \mathbf{u}^{(1)}$ is relative vorticity, $\mathbf{u}^{(1)} = (u^{(1)}, v^{(1)})$ is horizontal velocity at first order, $S_\zeta = \mathbf{e}_r \cdot \nabla_s \times \mathbf{S}_u^{(3)}$, and $w^{(4)}$ is known from the $\mathscr{O}(\varepsilon^6)$ thermodynamic equation (A11)

$$w^{(4)} = -\frac{1}{\partial\theta^{(2)}/\partial z}\left[\frac{\partial\theta^{(3)}}{\partial t_p} + \frac{\partial\theta^{(4)}}{\partial t_s} + \mathbf{u}^{(0)}\cdot\nabla_p\theta^{(3)} + \mathbf{u}^{(0)}\cdot\nabla_s\theta^{(4)} + \mathbf{u}^{(1)}\cdot\nabla_p\theta^{(2)} - S_\theta^{(6)}\right] \quad (5)$$

where $\nabla_p = \left((a\cos\phi_p)^{-1}\partial/\partial\lambda_p, a^{-1}\partial/\partial\phi_p\right)$. Substituting (5) into (4) gives

$$\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\left[\frac{\partial\theta^{(3)}}{\partial t_p} + \frac{\partial\theta^{(4)}}{\partial t_s} + \mathbf{u}^{(0)}\cdot\nabla_p\theta^{(3)} + \mathbf{u}^{(0)}\cdot\nabla_s\theta^{(4)} + \mathbf{u}^{(1)}\cdot\nabla_p\theta^{(2)} - S_\theta^{(6)}\right]\right) + \frac{\partial}{\partial t_s}\zeta^{(1)} + \mathbf{u}^{(0)}\cdot\nabla_s\zeta^{(1)} + \beta v^{(1)} = S_\zeta. \quad (6)$$

The first term in brackets on the left-hand-side of (6) can be simplified. Firstly notice that $\rho^{(0)}$, $\theta^{(2)}$ and f do not depend on t_s, thus $\partial/\partial t_s$ can be brought outside the brackets. The other terms in the first term can be simplified using thermal wind balance (A9a, A9b). This leads to cancellation of terms with $\partial u^{(0)}/\partial z$, $\partial\mathbf{u}_s^{(1)}/\partial z$, or $\partial\mathbf{u}_p^{(1)}/\partial z$ (with $\mathbf{u}_p^{(1)}$ and $\mathbf{u}_s^{(1)}$ as the horizontal velocities for planetary and synoptic scales, respectively), which means that velocities can be taken out of the $\partial/\partial z$ derivative. This yields the potential vorticity equation

$$\left(\frac{\partial}{\partial t_s} + u_m^{(0)}\frac{1}{a\cos\phi_p}\frac{\partial}{\partial\lambda_s}\right)q_s^{(4)} + \left(\frac{\partial}{\partial t_p} + u_m^{(0)}\frac{1}{a\cos\phi_p}\frac{\partial}{\partial\lambda_p}\right)q_p^{(3)} + (v_s^{(1)} + v_p^{(1)})\hat{\beta} = S^{PV} \quad (7)$$

where

$$q_s^{(4)}(\mathbf{X}_p,\mathbf{X}_s,z) = \frac{1}{f}\nabla_s^2\pi^{(4)} + \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}\theta^{(4)}}{\partial\theta^{(2)}/\partial z}\right), \tag{8a}$$

$$q_p^{(3)}(\mathbf{X}_p,z) = \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}\theta^{(3)}}{\partial\theta^{(2)}/\partial z}\right), \tag{8b}$$

$$\hat{\beta}(\phi_p,t_m,z) = \beta + \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\frac{\partial}{a\partial\phi_p}\left(\rho^{(0)}\theta^{(2)}\right)}{\partial\theta^{(2)}/\partial z}\right), \tag{8c}$$

$$S_p^{PV} = \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}\overline{S_\theta^{(6)}}^{x_s,t_s,y_s}}{\partial\theta^{(2)}/\partial z}\right), \tag{8d}$$

$$S_s^{PV} = \mathbf{e}_r\cdot\nabla_s\times\mathbf{S}_u^{(3)} + \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}\left(S_\theta^{(6)} - \overline{S_\theta^{(6)}}^{x_s,t_s,y_s}\right)}{\partial\theta^{(2)}/\partial z}\right), \tag{8e}$$

$S^{PV} = S_s^{PV} + S_p^{PV}$, $u_m^{(0)} = u^{(0)}$ is the zonal velocity of the zonal mean flow, here $\theta^{(3)}$ and $\theta^{(4)}$ correspond to planetary and synoptic scale potential temperature, respectively, $\theta^{(2)}$ is the leading order potential temperature of the mean flow, $\pi^{(i)} = p^{(i)}/\rho^{(0)}$, $\theta^{(i=2,3,4)} = \partial\pi^{(i=2,3,4)}/\partial z$, $q_p^{(3)}$ is planetary scale PV, $q_s^{(4)}$ is synoptic scale PV, $\hat{\beta}$ is the effective background PV gradient, $\zeta^{(1)} = f^{-1}\nabla_s^2\pi^{(4)}$ is relative vorticity on the synoptic scale, and S^{PV}, S_s^{PV} and S_p^{PV} represent the source-sink terms for the full PV, synoptic scale PV and planetary scale PV, respectively. A similar equation to (7) can be obtained by linearising (A5) in DK13, though without the planetary scale PV as it is then absorbed in the background PV gradient as the zonal mean flow. Similarly, (9) below can be linked to (44) in DK13.

Equation (7) can then be split into planetary and synoptic PV equations, by averaging over synoptic scales: only the planetary scale terms remain, and the residual represents the synoptic scale equation (DK13). This yields

$$\left(\frac{\partial}{\partial t_s} + u_m^{(0)}\frac{1}{a\cos\phi_p}\frac{\partial}{\partial\lambda_s}\right)q_s^{(4)} + v_s^{(1)}\hat{\beta} = S_s^{PV} \tag{9}$$

for synoptic scales, and

$$\left(\frac{\partial}{\partial t_p}+u_m^{(0)}\frac{1}{a\cos\phi_p}\frac{\partial}{\partial\lambda_p}\right)q_p^{(3)}+v_p^{(1)}\hat{\beta}=S_p^{PV} \tag{10}$$

for planetary scales. The synoptic scale PV equation (9) closely resembles the QG PV equation, with the main differences arising in the background PV gradient.

The background PV gradient $\hat{\beta}$ resembles the background PV gradient used in Charney's baroclinic instability model (e.g. Hoskins and James 2014). However, in Charney's model (and also in the QG model) there is no dependence of the static stability N^2 (linked to background potential temperature) on latitude (ϕ_p), as there is here since $\theta^{(2)}=\theta^{(2)}(\phi_p,t_m,z)$. The QG background PV gradient, on the other hand, includes the mean flow relative vorticity gradient ($-\partial^2 u_m^{(0)}/\partial\phi_p^2$), which is not present here due to the planetary scaling. This means that $\hat{\beta}$ represents planetary geostrophy (e.g. Phillips 1963, DK09), but it is more realistic than in QG due to the dependence of background PV gradient on latitude.

The planetary scale PV equation (10) also resembles the QG PV equation, however the planetary scale PV (8b) only includes the stretching term (again due to the planetary scaling we chose). Note that the planetary and synoptic scale PV equations are independent of each other in this small amplitude limit, which implies no direct interaction between planetary and synoptic scales — their interaction only occurs via source-sink terms, the mean flow, or at higher order. This independence is not present in DK13's finite amplitude theory where the synoptic and planetary scale waves interact at leading order.

This analysis suggests that the QG approximation can be used locally for both planetary and synoptic scale PV. Note, however, that this is only true in this small amplitude case (in the finite amplitude theory of DK13 this approach is not applicable for the planetary scales).

The potential vorticity equation can be written in a different form (the one used in DK13 for the planetary scale), with a vertical advection term in the PV equation, starting from (6). Following the derivations in DK09 and DK13, we get

$$\frac{\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\left[\left(\mathbf{u}^{(1)}\cdot\nabla_m + w^{(4)}\frac{\partial}{\partial z}\right)q_m^{(2)} + \left(\frac{\partial}{\partial t_s} + \mathbf{u}_m^{(0)}\cdot\nabla_s\right)q_{s,2}^{(4)} + \left(\frac{\partial}{\partial t_p} + \mathbf{u}_m^{(0)}\cdot\nabla_p\right)q_{p,2}^{(3)}\right] = S^{PV2} \tag{11}$$

where

$$q_{s,2}^{(4)} = \frac{\zeta^{(1)}}{\rho^{(0)}}\frac{\partial\theta^{(2)}}{\partial z} + \frac{f}{\rho^{(0)}}\frac{\partial\theta^{(4)}}{\partial z},$$

$$q_{p,2}^{(3)} = \frac{f}{\rho^{(0)}}\frac{\partial\theta^{(3)}}{\partial z},$$

$$q_m^{(2)} = \frac{f}{\rho^{(0)}}\frac{\partial\theta^{(2)}}{\partial z}, \text{ and}$$

$$S^{PV2} = S_\zeta + \frac{f}{\partial\theta^{(2)}/\partial z}\frac{\partial S_\theta^{(6)}}{\partial z}.$$

Here $q_{s,2}^{(4)}$, $q_{p,2}^{(3)}$, $q_m^{(2)}$, and S^{PV2} are the DK synoptic, planetary and mean flow PVs, and the corresponding PV source term, respectively.

The PV equation (11) is closely related to the Ertel PV equation. However, it includes vertical advection, which is problematic with respect to obtaining a QG wave activity equation. As shown in (7) we can eliminate the vertical advection term by including it in the stretching term of the synoptic or planetary scale PV. This is similar to the classical QG approximation of Charney and Stern (1962), in which they point out that the QG PV equation is the QG approximation to the PV equation, however the QG PV is not the QG approximation to the Ertel PV (because the QG PV equation only includes horizontal advection). Notice that in (11) there is also the mean flow PV, whereas equation (7) only has the background PV gradient that came from this mean flow PV (but not via the direct meridional derivative of $q_m^{(2)}$, i.e. $\hat{\beta} \neq \partial q_m^{(2)}/\partial y_p$). This means that the

QG approximation of PV would not work for the zonal mean flow, which is consistent with the arguments above on the relation between the QG PV and the Ertel PV.

4. Wave activity equation and the equations for the mean flow

a. Wave activity equation

Wave activity is a quantity that is quadratic in amplitude and is conserved in the absence of forcing and dissipation (e.g. Vallis 2006). To derive an equation for wave activity, known as the Eliassen-Palm (EP) relation, we multiply the PV equations (9) and (10) by $q_s^{(4)}$ and $q_p^{(3)}$, respectively, and divide them by $\hat{\beta}$ (as done in e.g. Plumb 1985). This yields

$$\frac{\partial \mathscr{A}_s}{\partial t_s} + \nabla_s^{3D} \cdot \mathbf{F}_s = S_s^{wa} \tag{12}$$

$$\frac{\partial \mathscr{A}_p}{\partial t_p} + \nabla_p^{3D} \cdot \mathbf{F}_p = S_p^{wa} \tag{13}$$

where

$$\mathscr{A}_s = \frac{\rho^{(0)} q_s^{(4)^2}}{2\hat{\beta}},$$

$$\mathscr{A}_p = \frac{\rho^{(0)} q_p^{(3)^2}}{2\hat{\beta}}$$

are synoptic and planetary scale wave activities, respectively, $S_s^{wa} = S_s^{PV} \rho^{(0)} q_s^{(4)}/\hat{\beta}$ and $S_p^{wa} = S_p^{PV} \rho^{(0)} q_p^{(3)}/\hat{\beta}$ are wave activity source-sink terms,

$$\mathbf{F}_s = \left(u_m^{(0)} \mathscr{A}_s + \frac{\rho^{(0)}}{2} \left(v_s^{(1)^2} - u_s^{(1)^2} - \frac{\theta^{(4)^2}}{\partial \theta^{(2)}/\partial z} \right), -\rho^{(0)} v_s^{(1)} u_s^{(1)}, \rho^{(0)} f \frac{v_s^{(1)} \theta^{(4)}}{\partial \theta^{(2)}/\partial z} \right),$$

$$\mathbf{F}_p = \left(u_m^{(0)} \mathscr{A}_p - \frac{\rho^{(0)}}{2} \frac{\theta^{(3)^2}}{\partial \theta^{(2)}/\partial z}, 0, \rho^{(0)} f \frac{v_p^{(1)} \theta^{(3)}}{\partial \theta^{(2)}/\partial z} \right)$$

are synoptic and planetary Eliassen-Palm (EP) fluxes, respectively, and ∇^{3D} means that the gradient includes the vertical derivative.

Note how the planetary scale EP flux does not have a meridional component (no momentum flux), and that the synoptic scale EP flux closely resembles Plumb (1985)'s total flux $\mathbf{B}^{(T)}$, with the main difference, again, arising in $\hat{\beta}$. Also, $u_s^{(1)}$ is actually composed of $u_s^{(1)} = [u]_s^{(1)} + u_s^{*(1)}$ (with [.] as zonal mean and * as perturbation from zonal mean), which is another difference to Plumb's $\mathbf{B}^{(T)}$ flux.

We can also relate these expressions to Hoskins et al. (1983)'s E-vector, where the difference is in the zonal component of the E-vector, which lacks the wave activity advection ($[u]\mathscr{A}$) and potential temperature ($\propto -\theta^{*2}$) terms.

Nonetheless, the synoptic scale EP flux is similar to the QG form of EP flux (e.g. Edmon et al. 1980), especially if zonally averaged. The planetary scale wave activity implies that the momentum fluxes and hence barotropic processes at those scales are less important than heat fluxes and baroclinic processes. Also, this emphasises the fact that planetary and synoptic scales do not interact directly, but rather through other processes (source-sink terms or the mean flow) as the two wave activity equations are at different orders and have no "cross" terms. The wave activity equations are at different orders as the planetary (10) and synoptic (9) PV equations are multiplied by $q_p^{(3)}$ and $q_s^{(4)}$, respectively, which are of different orders. This is because they have different horizontal derivatives associated with them (q_s has synoptic and q_p has planetary).

Averaging over synoptic scales (λ_s, ϕ_s, t_s; denoted by overline and s) in (12) and over planetary scales (λ_p, t_p; denoted by overline and p) in (13) gives

$$\frac{\partial}{\partial z}\left(\rho^{(0)} f \frac{\overline{v_s^{(1)}\theta^{(4)}}^s}{\partial\theta^{(2)}/\partial z}\right) = \overline{S_s^{wa}}^s \approx -r_s\overline{\mathscr{A}_s}^s \tag{14}$$

$$\frac{\partial}{\partial z}\left(\rho^{(0)} f \frac{\overline{v_p^{(1)}\theta^{(3)}}^p}{\partial\theta^{(2)}/\partial z}\right) = \overline{S_p^{wa}}^p \approx -r_p\overline{\mathscr{A}_p}^p \tag{15}$$

where $r_{s,p}$ are effective damping coefficients. Note that the approximation $\overline{S^{wa}_{s,p}}^{s,p} \approx -r_{s,p}\overline{\mathscr{A}_{s,p}}^{s,p}$ does not follow from the equations themselves, but is a heuristic relation used as a device to help us better understand the physical interpretation of the equations. These equations imply that under these averages both synoptic and planetary scale wave activities change via heat flux terms on timescales longer than t_s or t_p (as we averaged over those) - e.g. timescale $\varepsilon^4 t$ (between t_p and t_m) or t_m. Averaging only over the zonal and time dimensions, the synoptic scale wave activity would still be influenced by the synoptic scale momentum fluxes.

b. Barotropic equation

As the wave activity equation represents the equation for the eddies, we need additional equations for the mean flow to get the influence from the eddies on the mean flow. The barotropic pressure equation is derived (following DK13) from the $\mathscr{O}(\varepsilon^5)$ momentum equation (A8) using the relevant thermodynamic, hydrostatic, thermal wind, momentum and continuity equations averaged not only over t_s, λ_s, ϕ_s and t_p, λ_p, but also over z (denoted by overline and z). This yields momentum equation (B6) (see Appendix B for more details), which can be used to derive the barotropic pressure equation, taking $\partial/\partial\tilde{y}_p$ of (B6), eliminating the term $\partial\left(\overline{v^{(4)}\rho^{(0)}}^{s,p,z}\right)/\partial\tilde{y}_p$ via (B5), multiplying it by f and recalling (A4):

$$\frac{\partial}{\partial t_m}\left(\frac{\partial}{\partial\tilde{y}_p}\frac{1}{f}\frac{\partial}{\partial y_p}\overline{p^{(2)}}^{s,p,z} - \frac{\beta}{f^2}\frac{\partial}{\partial y_p}\overline{p^{(2)}}^{s,p,z} - f\overline{p^{(2)}}^{s,p,z}\right) - \frac{\partial}{\partial\tilde{y}_p}N_1 + \frac{\beta}{f}N_1 - fN_2 = -S_{barotropic} \tag{16}$$

with

$$N_1 = \frac{\partial}{\partial\tilde{y}_p}\overline{\left(\rho^{(0)}\underline{\underline{v_p^{(1)}u_p^{(1)}}} + \rho^{(0)}\underline{\underline{v_s^{(1)}u_s^{(1)}}}\right)}^{s,p,z} - \frac{\tan\phi_p}{a}\overline{\left(\rho^{(0)}\underline{\underline{v_p^{(1)}u_p^{(1)}}} + \rho^{(0)}\underline{\underline{v_s^{(1)}u_s^{(1)}}}\right)}^{s,p,z},$$

$$N_2 = \frac{\partial}{\partial\tilde{y}_p}\left(\overline{\rho^{(0)}\underline{\underline{v_p^{(1)}\theta^{(3)}}}}^{s,p,z}\right),$$

$$S_{barotropic} = \overline{f\rho^{(0)}S_\theta^{(7)}}^{s,p,z} + f\frac{\partial}{\partial\tilde{y}_p}\left(\overline{\left(\rho^{(2)}+\rho^{(0)}\theta^{(2)}\right)\frac{S_u^{(3)}}{f}}^{s,p,z}\right)$$
$$+\left(\frac{\partial}{\partial\tilde{y}_p}-\frac{\beta}{f}\right)\left[\overline{\rho^{(0)}S_u^{(5)}}^{s,p,z} + \left\{\frac{\partial}{\partial\tilde{y}_p}-\frac{\tan\phi_p}{a}\right\}\left(\overline{\frac{S_u^{(3)}}{f}u^{(0)}\rho^{(0)}}^{s,p,z}\right) - \frac{\rho^{(0)}\overline{S_\theta^{(6)}}^{s,p,z}\cos\phi_p}{f\partial\theta^{(2)}/\partial z}\right]$$

where the underlined terms represent eddy forcing of the mean flow, $\partial/\partial\tilde{y}_p \equiv (a\cos\phi_p)^{-1}\partial\cos\phi_p/\partial\phi_p$, and $\partial/\partial y_p \equiv a^{-1}\partial/\partial\phi_p$. This evolution equation (16) for $p^{(2)}$ on the t_m scale is similar to DK13's $p^{(2)}$ evolution on the t_p scale when no source terms are considered. Using geostrophic balance for $u^{(0)}$, (16) can be rewritten as

$$\left(\frac{\partial}{\partial\tilde{y}_p}-\frac{\beta}{f}\right)\frac{\partial\overline{\rho^{(0)}u^{(0)}}^{s,p,z}}{\partial t_m} + f\frac{\partial\overline{p^{(2)}}^{s,p,z}}{\partial t_m} + \left(\frac{\partial}{\partial\tilde{y}_p}-\frac{\beta}{f}\right)N_1 + fN_2 = S_{barotropic}. \tag{17}$$

This equation implies that although both the synoptic and planetary scale momentum fluxes affect the barotropic part of the mean flow, only the planetary scale heat fluxes N_2 are relevant.

The zonal mean flow equations at different orders can be further written in TEM form (Andrews and McIntyre 1976; Edmon et al. 1980), from which a non-acceleration theorem can be derived using the wave activity equations. This is addressed in Appendix D. Note that an evolution equation for $p^{(3)}$ can also be derived, however under the $\lambda_p, \lambda_s, t_s, \phi_s, z$ average it only evolves through diabatic and frictional processes (D9).

c. Baroclinic equation

The barotropic equation (17) shows how barotropic processes affect the zonal mean flow, however we are also interested in the baroclinic processes. Therefore, a baroclinic equation for the zonal mean flow (i.e. equation for baroclinicity $\propto \partial u^{(0)}/\partial z$) is derived from the $\mathscr{O}(\varepsilon^7)$ thermodynamic equation (A12), using the relevant continuity and momentum equations averaged over t_s, λ_s, t_p, λ_p (denoted with overline), and taking $\partial/\partial y_p$ of the resulting equation (B7b). The relevant equations (and their derivations) are given in Appendix B, hence using (B10-B14) yields:

$$-\frac{\partial}{\partial t_m}\left(\overline{f\rho^{(0)}\frac{\partial u^{(0)}}{\partial z}}^{\lambda_s,t_s,p}\right)+\frac{\partial}{\partial y_p}\left[\frac{\partial}{\partial \tilde{y}_p}\left(\overline{v_p^{(1)}\rho^{(0)}\theta^{(3)}}^{\lambda_s,t_s,p}\right)+\frac{\partial}{\partial \tilde{y}_s}\left(\overline{v_s^{(1)}\rho^{(0)}\theta^{(4)}}^{\lambda_s,t_s,p}\right)\right]$$
$$-\frac{\partial}{\partial y_p}\left[\frac{\partial}{\partial z}\left(\overline{v_p^{(1)}\rho^{(0)}\theta^{(3)}}^{\lambda_s,t_s,p}\frac{\partial\theta^{(2)}/\partial y_p}{\partial\theta^{(2)}/\partial z}\right)-\overline{\rho^{(0)}u_s^{(1)}\frac{\partial\theta^{(3)}}{\partial x_p}}^{\lambda_s,t_s,p}\right]$$
$$-\frac{\partial}{\partial y_p}\left[\frac{\partial\theta^{(2)}}{\partial z}\int_0^{z_{max}}\rho^{(0)}\frac{\partial}{\partial\tilde{y}_s}\left(\frac{\partial}{\partial\tilde{y}_s}\left(\frac{\overline{v_s^{(1)}u_s^{(1)}}^{\lambda_s,t_s,p}}{f}\right)\right)\mathrm{d}z\right]=S_{baroclinic}\quad(18)$$

with

$$S_{baroclinic}=\frac{\partial}{\partial y_p}\left[\frac{\partial}{\partial\tilde{y}_s}\left(\overline{\frac{S_u^{(3)}}{f}\rho^{(0)}\theta^{(3)}}^{\lambda_s,t_s,p}\right)-\frac{\rho^{(0)}\overline{\theta^{(3)}S_\theta^{(6)}}^{\lambda_s,t_s,p}}{\partial\theta^{(2)}/\partial z}+\rho^{(0)}\frac{\overline{S_u^{(3)}}^{\lambda_s,t_s,p}}{f}\frac{\partial\theta^{(2)}}{\partial y_p}\right]$$
$$+\frac{\partial}{\partial y_p}\left[\overline{S_\theta^{(7)}\rho^{(0)}}^{\lambda_s,t_s,p}-S_{w5}\frac{\partial\theta^{(2)}}{\partial z}-\frac{\partial}{\partial\tilde{y}_s}\left(\frac{z}{a}\frac{\overline{S_u^{(3)}}^{\lambda_s,t_s,p}}{f}\right)+\frac{\partial}{\partial z}\left(\frac{z}{a}\frac{\overline{S_\theta^{(6)}}^{\lambda_s,t_s,p}}{\partial\theta^{(2)}/\partial z}\right)\right],$$

$$S_{w5}=-\int_0^{z_{max}}\left[\frac{\partial}{\partial\tilde{y}_s}\left(\rho^{(0)}\left\{\frac{\overline{S_\theta^{(6)}}^{\lambda_s,t_s,p}}{f}\frac{\partial u^{(0)}/\partial z}{\partial\theta^{(2)}/\partial z}-\frac{\overline{S_u^{(4)}}^{\lambda_s,t_s,p}}{f}\right\}\right)-\frac{\partial}{\partial\tilde{y}_p}\left(\rho^{(0)}\frac{\overline{S_u^{(3)}}^{\lambda_s,t_s,p}}{f}\right)\right]\mathrm{d}z,$$

where the terms with z/a come from corrections to the shallow-atmosphere approximation of the thermodynamic and continuity equations. Averaging (18) over the synoptic meridional scale (ϕ_s) gives

$$-\frac{\partial}{\partial t_m}\left(\overline{f\rho^{(0)}\frac{\partial u^{(0)}}{\partial z}}^{s,p}\right)+\frac{\partial}{\partial y_p}\left[\frac{\partial}{\partial\tilde{y}_p}\left(\overline{v_p^{(1)}\rho^{(0)}\theta^{(3)}}^{s,p}\right)-\frac{\partial}{\partial z}\left(\overline{v_p^{(1)}\rho^{(0)}\theta^{(3)}}^{s,p}\frac{\partial\theta^{(2)}/\partial y_p}{\partial\theta^{(2)}/\partial z}\right)\right]$$
$$=\overline{S}^{\phi_s}_{baroclinic}\quad(19)$$

which implies that baroclinicity is not affected by the synoptic scale heat fluxes $(\rho^{(0)}v_s^{(1)}\theta^{(4)})$, only by baroclinic source terms ($S_{baroclinic}$) and planetary scale heat fluxes $(\rho^{(0)}v_p^{(1)}\theta^{(3)})$. The absence of a synoptic scale heat flux contribution to the baroclinicity tendency is discussed in section 6.

5. Angular momentum conservation

Apart from the mean flow equations (baroclinic and barotropic) and the eddy equations (wave activity), angular momentum conservation provides additional information about the transfer of angular momentum between the earth and the atmosphere, which has implications for the surface easterlies in the tropics and westerlies in the midlatitudes (e.g. Holton 2004). Hence, it is important to show that such a budget can be found also in the asymptotic model.

Generally, the angular momentum for the hydrostatic primitive equations takes the form (e.g. Holton 2004)

$$M = au\cos\phi + a^2\Omega\cos^2\phi \tag{20}$$

where a is the radius of the Earth, Ω is the Earth's rotation rate, ϕ is meridional coordinate, u is zonal velocity, and M is angular momentum per unit mass.

In the asymptotic regime, a nondimensional version of angular momentum must be used. To derive the nondimensional version of (20), define nondimensional terms (similarly as in section 2): $u = u^* u_{ref}$, $a = a^*\varepsilon^{-3}h_{sc}$, $\Omega = \frac{1}{2}\Omega^*(2\Omega_{ref})$ and $M = M^* u_{ref} h_{sc}\varepsilon^{-3}$, where u_{ref} and h_{sc} were defined in section 2, Ω_{ref} is the Earth's rotation rate (previously denoted Ω), $M \propto \varepsilon^{-3}$ as it needs to be of the same order as other terms, and the asterisk (*) denotes nondimensional parameters. Now divide (20) by $u_{ref}h_{sc}$ to get nondimensional angular momentum

$$\varepsilon^{-3}M^* = a^*\varepsilon^{-3}u^*\frac{u_{ref}h_{sc}}{u_{ref}h_{sc}}\cos\phi + (\varepsilon^{-3})^2(a^*)^2\frac{1}{2}\Omega^*\frac{h_{sc}}{h_{sc}}\frac{h_{sc}2\Omega_{ref}}{u_{ref}}\cos^2\phi. \tag{21}$$

Cancelling out a few terms, setting Ω^* to unity, recognising that[4] $h_{sc}2\Omega_{ref}/u_{ref} = Ro^{-1} \approx \varepsilon$, and omitting asterisks for simplicity, yields the nondimensional angular momentum

$$\varepsilon^{-3}M = \varepsilon^{-3}au\cos\phi + \varepsilon^{-3}\varepsilon^{-2}\frac{1}{2}a^2\cos^2\phi. \tag{22}$$

[4] Here the Rossby number used is the same as the one defined in DK09, DK13: $Ro^{-1} \approx Ro_{QG} \approx \varepsilon$.

Taking the total derivative (2) of M in (22) gives the nondimensional angular momentum equation

$$\varepsilon^{-3}\frac{\mathrm{D}M}{\mathrm{D}t} = \varepsilon^{-3} a\cos\phi\frac{\mathrm{D}u}{\mathrm{D}t} - uv\sin\phi - \varepsilon^{-2} afv\cos\phi \tag{23}$$

using $\partial/\partial t = \varepsilon^5\partial/\partial t_m$, $w^{(0)} = w^{(1)} = w^{(2)} = w^{(3)} = 0$ (as derived in Appendix A), and all parameters are nondimensional. Notice that

$$\frac{\partial\cos^2\phi}{\partial\phi} = -2\cos\phi\sin\phi,$$

which means that the factor 2 from this equation cancels out the factor $1/2$ in M (22). Here

$$v = \varepsilon^{-3} a\frac{D\phi}{Dt} = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + ...,$$

$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + ...$$

The angular momentum equation and its conservation for the zonal mean flow ($u^{(0)}$) are derived in Appendix C. The second order angular momentum equation is

$$\begin{aligned}\rho\frac{\mathrm{D}M}{\mathrm{D}t_m} = a\cos\phi_p\rho^{(0)}\frac{\mathrm{D}u^{(0)}}{\mathrm{D}t_m} - (\rho^{(0)}u^{(1)}v^{(1)} + \rho^{(0)}u^{(0)}v^{(2)})\sin\phi_p \\ -f(\rho^{(0)}v^{(4)} + \rho^{(2)}v^{(2)} + \rho^{(3)}v^{(1)})a\cos\phi_p,\end{aligned} \tag{24}$$

from which it is shown (Appendix C) that M is conserved (using the 5^{th} order momentum equation A8) in the absence of source-sink terms and orography, yielding

$$\iiint_{Vp}\frac{\partial\overline{(\rho M)^{(2)}}^{s,t_p}}{\partial t_m}\mathrm{d}V_p = 0 \tag{25}$$

where V_p is volume on planetary scales (λ_p, ϕ_p, z).

The barotropic pressure equation (17) can now be rewritten using the angular momentum equation (Appendix C) as

$$\left(\frac{\partial}{\partial\tilde{y}_p} - \frac{\beta}{f}\right)\left\{\overline{\frac{\rho}{a\cos\phi_p}\frac{DM}{Dt_m}}^{s,p,z}\right\} - f\frac{\partial\overline{\rho^{(2)}}^{s,p,z}}{\partial t_m} = -f\frac{\partial\overline{p^{(2)}}^{s,p,z}}{\partial t_m} - f\frac{\partial}{\partial\tilde{y}_p}\left(\overline{\rho^{(0)}v_p^{(1)}\theta^{(3)}}^{s,p,z}\right) \tag{26}$$

where the overbar denotes average over $t_s, t_p, \lambda_s, \lambda_p, \phi_s, z$. This shows that the two quantities are directly linked.

Note that the surface pressure tendency $\partial \overline{p^{(2)}}^{s,p,z}/\partial t_m$ in (17) and (26) reflects the response of planetary angular momentum to an imposed torque, via mass redistribution, and is an essential component of the angular momentum equation at planetary scales (Haynes and Shepherd 1989). The present analysis has shown further that the planetary-scale meridional heat flux contributes to this meridional mass redistribution. That the synoptic scale heat flux does not so contribute can be anticipated from the scaling arguments of Haynes and Shepherd (1989).

6. The zonally homogeneous case

If there are no forced planetary scale waves in the system, then there is no justification for separate λ_p and t_p scales. If the zonal and synoptic scale (including ϕ_s) average is taken in such a case, then the wave activity, barotropic and baroclinic equations become:

$$\frac{\partial}{\partial z}\left(\rho^{(0)} f \frac{\overline{v_s^{(1)}\theta^{(4)}}^{s}}{\partial \theta^{(2)}/\partial z}\right) \approx -r_s \overline{\mathscr{A}_s}^{s}, \tag{27a}$$

$$\left(\frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f}\right)\frac{\partial \overline{\rho^{(0)} u^{(0)}}^{s,p,z}}{\partial t_m} - f\frac{\partial \overline{p^{(2)}}^{s,p,z}}{\partial t_m} + \left(\frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f}\right)\overline{N_1}^{s,p,z} = \overline{S_{barotropic}}^{s,p,z}, \tag{27b}$$

$$-\frac{\partial}{\partial t_m}\left(\overline{f\rho^{(0)}\frac{\partial u^{(0)}}{\partial z}}^{s,p}\right) = \overline{S_{baroclinic}}^{s,p}. \tag{27c}$$

These equations imply that under synoptic scale averaging, and to leading order, the wave activity is only affected by the heat fluxes through a quasi-steady balance, the barotropic part of the zonal mean flow tendency is only affected by the momentum fluxes (in N_1), and the baroclinicity tendency is only affected by source-sink terms. The latter can, however, be related to the source-sink terms in the wave activity and barotropic pressure equations. The most surprising of these relations are (27a) and (27c), which depend crucially on the averaging over ϕ_s. When the equations are not averaged over ϕ_s, then momentum fluxes appear in the wave activity equation and heat fluxes appear in the baroclinicity tendency equation.

These findings may help explain the empirical results of Thompson and Woodworth (2014), who found that the barotropic and baroclinic parts of the Southern Hemisphere (SH) flow variability were decoupled, with the barotropic part of the flow (characterised by the Southern Annular Mode (SAM), based on zonal mean zonal wind) being only affected by the momentum fluxes, and the baroclinic part of the flow (characterised by the baroclinic annular mode (BAM), based on EKE) being only affected by the heat fluxes. We assume here that the wave activity is closely linked to EKE. Indeed, Wang and Nakamura (2015, 2016) found that wave activity during the SH summer is only affected by the heat fluxes under an average over a few latitudinal bands (approximately 10^{o}), giving an equation similar to (27a). Here we put this view into a self-consistent mathematical perspective.

In a separate study, Thompson and Barnes (2014) found an oscillating relationship between the EKE and the heat fluxes with a timescale of 20-30 days. In their model, baroclinicity is affected by synoptic scale heat fluxes, through the assumption that

$$\frac{\partial^2 [v^* T^*]}{\partial y^2} = -l^2 [v^* T^*],$$

where l is meridional wave number, T is temperature, $[.]$ represents zonal mean and asterisk $(^*)$ represents perturbations therefrom. This relation is not present here due to the chosen scaling and the averaging over synoptic scales. Equation (18) does in fact have the heat fluxes, acting on synoptic scales, which due to the sublinear growth condition (DK13) disappear in (27c), as mentioned above.

Pfeffer (1987, 1992) argued that heat fluxes (vertical EP fluxes) grow in the part of the domain with low stratification parameter S. Pfeffer's S can be related to ε as $S = (L_R/a^*)^2 \approx \varepsilon^2$, where

$L_R \approx \varepsilon a^*$ is Rossby deformation radius (a typical synoptic scale) and a^* is Earth's radius (a typical planetary scale). Since here we consider the case with $\varepsilon \ll 1$, we are then in a regime where $S \ll 1$ and hence the heat fluxes act to drive the residual meridional circulation rather than the zonal mean flow, and the vertical derivative of the zonal mean flow (i.e. baroclinicity) is not related to EP flux divergence to leading order (see equations (6)-(9) in Pfeffer 1992). This suggests a barotropic response of the zonal mean flow to eddy fluxes after averaging over synoptic scales, which is consistent with (27b) and (27c).

Zurita-Gotor (2017) showed further that there is a low frequency suppression of heat fluxes (at periods longer than 20-30 days) and concluded that at longer timescales (considered here) the meridional circulation and diabatic processes are more important for the baroclinicity than the synoptic scale heat fluxes (consistent with (27c)).

7. Conclusions

In this paper we have provided a theoretical framework for planetary-synoptic-zonal mean flow interactions in the small amplitude limit with a scale separation in the meridional direction, as well as in the zonal direction, between planetary and synoptic scales. Thus the synoptic scale eddies are assumed to be isotropic (which is the case also in QG theory). These assumptions allow us to derive strong results, e.g. a lack of direct interaction between the planetary and synoptic waves, and a lack of a direct link between the baroclinic and barotropic components of the flow when only synoptic scale fluxes are considered.

We derived planetary and synoptic scale PV equations (9, 10), and equations for the eddies (wave activity equations (14-15)), the barotropic part of the zonal mean flow (17) and the baroclinic part of the zonal mean flow (19). A crucial step in deriving these equations was finding a form of the PV equation that eliminated the effect of vertical advection. The synoptic scale PV then resembled QG PV and the planetary PV resembled that of planetary geostrophy, i.e. with only stretching vorticity representing PV on planetary scales (e.g. Phillips 1963). These equations provide an alternative view to the conventional Transformed Eulerian Mean (TEM) framework (first introduced in Andrews and McIntyre 1976), which combines all components into two equations that are linked through the Eliassen-Palm flux.

The background PV gradient (8c) that emerged from the equations lacks the relative vorticity term as in planetary geostrophy (Phillips 1963), implying the dominance of baroclinic processes for eddy generation. Thus this PV gradient resembles that of Charney's baroclinic instability model (e.g. Hoskins and James 2014), but is more general as it includes variations in static stability in both the vertical and meridional directions. The latter should be stressed as this is the main difference to QG dynamics in this model.

In terms of the baroclinic life cycle (Simmons and Hoskins 1978), the barotropic pressure equation (17) would be relevant in the breaking region of the storm track and the baroclinic equation (19) would be more relevant in the source region. We also showed that only the planetary scale heat fluxes affect the baroclinicity (19), that both planetary and synoptic scale momentum fluxes, as well as planetary scale heat fluxes, affect the barotropic zonal mean flow (17), and that the planetary waves and synoptic scale eddies only interact via the zonal mean flow, the source-sink terms or at higher order approximations. Since both the barotropic (17) and baroclinic (19) parts of the zonal mean flow are affected by the planetary scale heat fluxes, the latter could provide a link between upstream and downstream development of storm tracks. The barotropic equation (17) was also directly linked to the angular momentum equation (26), which has not been noted in previous work. This linkage revealed the importance of planetary scale heat fluxes (via meridional mass transport) for the angular momentum budget (Haynes and Shepherd 1989).

The importance of planetary scale waves was also noted in Kaspi and Schneider (2011, 2013), who found that the termination of storm tracks downstream is related to stationary waves and the baroclinicity associated with them. Stationary waves are especially important locally in contributing to heat fluxes, which enhance temperature gradients upstream, and reduce them downstream.

When considering only the synoptic scale eddies (when planetary scale eddies are weak, as e.g. in aquaplanet simulations or in the Southern Hemisphere), we find that under synoptic scale averaging the barotropic zonal mean flow (27b) is only affected by the momentum fluxes, the baroclinicity (27c) is only affected by the source-sink terms, and wave activity (27a) is only related to heat fluxes (as in Thompson and Woodworth 2014). This suggests that the baroclinicity is primarily diabatically driven. Understanding the decoupling of the baroclinic and barotropic parts of the flow (in the case of weak planetary scale waves) is addressed in a companion study (Boljka et al. 2018), where it is shown that at timescales longer than synoptic the EKE is only affected by the heat fluxes and not momentum fluxes, confirming relation (27a).

As well as helping to understand a variety of previous results in the literature, one potential use of the theory presented here could be to help understand the barotropic response to climate change, which is fundamentally thermally driven. In general, we need a better understanding of the interaction between the baroclinic and barotropic parts of the flow, where planetary scale heat fluxes and diabatic processes may play an important role.

This theoretical framework could be extended by allowing finite amplitude eddies (as in DK13) and by relaxing the assumption of a separation of scales in latitude (e.g. Dolaptchiev 2008).

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APPENDIX A

The Multiscale Asymptotic Version of the Primitive Equations

Using the assumptions from section 2b the momentum, thermodynamic, continuity, hydrostatic and thermal wind balance equations at different orders ($\mathscr{O}(i)$) can be derived following DK09, DK13.

Hydrostatic balance

Up to 4th order:

$$\rho^{(i)} = -\frac{\partial p^{(i)}}{\partial z} \quad ; \quad i = 0, ..., 4. \tag{A1}$$

There is also a relationship between p and θ as defined in (47) in (DK09):

$$\frac{\partial \pi^{(i)}}{\partial z} = \theta^{(i)} \quad ; \quad i = 2, 3, 4 \tag{A2}$$

where $\pi^{(i)} = p^{(i)}/\rho^{(0)}$. This identity at the fourth order only holds if $\frac{\partial}{a\partial \phi_s}$ of θ is taken (and this relationship will only be used in this case).

Using (A2) and (A1) one gets a relationship between ρ, p and θ:

$$\rho^{(i)} = p^{(i)} - \rho^{(0)}\theta^{(i)} \quad ; \quad i = 2, 3 \tag{A3}$$

where an assumption is made that $\rho^{(0)} = \exp(-z)$.

Momentum equations

Below is the list of all momentum equations up to 5th order. Note that we derive the PV and wave activity equations from the 3rd order momentum equation, and we obtain a barotropic equation for the mean flow from the 5th order momentum equation.

$\mathscr{O}(\varepsilon^1)$ - geostrophic balance for zonal mean wind:

$$f\mathbf{e}_r \times \mathbf{u}^{(0)} = f\mathbf{e}_r \times \mathbf{u}_m^{(0)} = -\nabla_p \pi^{(2)} = -\frac{\partial}{\partial y_p}\pi^{(2)}\mathbf{e}_\phi \tag{A4}$$

where subscript m refers to the mean flow - $\mathbf{u}^{(0)}$ is related to the zonal mean zonal velocity. Note that $v^{(0)} = 0$.

$\mathscr{O}(\varepsilon^2)$ - geostrophic balance for 1st order wind (planetary and synoptic scale perturbations to zonal mean):

$$f\mathbf{e}_r \times \mathbf{u}^{(1)} = -\left(\nabla_p \pi^{(3)} + \nabla_s \pi^{(4)}\right) \tag{A5}$$

where $\mathbf{u}^{(1)} = \mathbf{u}_p^{(1)} + \mathbf{u}_s^{(1)}$ (with subscripts p and s referring to planetary and synoptic waves, respectively), such that $f\mathbf{e}_r \times \mathbf{u}_p^{(1)} = -\nabla_p \pi^{(3)}$ and $f\mathbf{e}_r \times \mathbf{u}_s^{(1)} = -\nabla_s \pi^{(4)}$.

$\mathscr{O}(\varepsilon^3)$ - first nontrivial order, used to derive PV equations:

$$\begin{aligned}\frac{\partial \mathbf{u}^{(1)}}{\partial t_s} + \mathbf{u}^{(0)} \cdot \nabla_s \mathbf{u}^{(1)} + f\mathbf{e}_r \times \mathbf{u}^{(2)} + \mathbf{e}_\phi \frac{u^{(0)}u^{(0)}\tan\phi_p}{a} &= \\ -\nabla_p \pi^{(4)} + \frac{\rho^{(2)}}{\rho^{(0)}}\nabla_p \pi^{(2)} - \nabla_s \pi^{(5)} + \mathbf{S}_u^{(3)}\end{aligned} \tag{A6}$$

$\mathscr{O}(\varepsilon^4)$ - we require only the u-momentum equation:

$$\begin{aligned}\frac{\partial u^{(2)}}{\partial t_s} + \frac{\partial u^{(1)}}{\partial t_p} + \mathbf{u}^{(1)} \cdot \nabla_s u^{(1)} + \frac{\partial}{\partial \tilde{x}_s}\left(u^{(0)}u^{(2)}\right) + \frac{\partial}{\partial \tilde{x}_p}\left(u^{(0)}u^{(1)}\right) + \\ v^{(1)}\frac{\partial}{\partial y_p}u^{(0)} + w^{(4)}\frac{\partial}{\partial z}u^{(0)} - fv^{(3)} - \frac{u^{(0)}v^{(1)}\tan\phi_p}{a} = \\ -\frac{\partial}{\partial x_p}\pi^{(5)} + \frac{\partial}{\partial x_p}\left(\frac{\rho^{(2)}}{\rho^{(0)}}\pi^{(3)}\right) - \frac{\partial}{\partial x_s}\pi^{(6)} + \frac{\partial}{\partial x_s}\left(\frac{\rho^{(2)}}{\rho^{(0)}}\pi^{(4)}\right) + S_u^{(4)}\end{aligned} \tag{A7}$$

$\mathscr{O}(\varepsilon^5)$ - again we require only the u-momentum equation, used to derive the barotropic pressure equation (equation for the zonal mean zonal flow):

$$\frac{\partial u^{(0)}}{\partial t_m}+\frac{\partial u^{(3)}}{\partial t_s}+\frac{\partial u^{(2)}}{\partial t_p}+\mathbf{u}^{(1)}\cdot\nabla_s u^{(2)}+\mathbf{u}^{(2)}\cdot\nabla_s u^{(1)}+\frac{\partial}{\partial\tilde{x}_s}\left(u^{(0)}u^{(3)}\right)+\frac{\partial}{\partial\tilde{x}_p}\left(u^{(0)}u^{(2)}\right)+$$
$$\mathbf{u}^{(1)}\cdot\nabla_p u^{(1)}+v^{(2)}\frac{\partial}{\partial y_p}u^{(0)}+w^{(4)}\frac{\partial}{\partial z}u^{(1)}+w^{(5)}\frac{\partial}{\partial z}u^{(0)}-fv^{(4)}$$
$$-\frac{u^{(0)}v^{(2)}\tan\phi_p}{a}-\frac{u^{(1)}v^{(1)}\tan\phi_p}{a}+w^{(4)}\cos\phi_p=-\frac{\partial}{\partial x_p}\pi^{(6)}+\frac{\partial}{\partial x_p}\left(\frac{\rho^{(2)}}{\rho^{(0)}}\pi^{(4)}\right)+\frac{\rho^{(3)}}{\rho^{(0)}}\frac{\partial}{\partial x_p}\pi^{(3)}-$$
$$\frac{\partial}{\partial x_s}\pi^{(7)}+\frac{\partial}{\partial x_s}\left(\frac{\rho^{(2)}}{\rho^{(0)}}\pi^{(5)}\right)+\frac{\partial}{\partial x_s}\left(\frac{\rho^{(3)}}{\rho^{(0)}}\pi^{(4)}\right)+S_u^{(5)}$$
(A8)

where in all equations $\frac{\partial}{\partial y_{p,s}}=\frac{1}{a}\frac{\partial}{\partial\phi_{p,s}}$, $\frac{\partial}{\partial\tilde{y}_{p,s}}=\frac{1}{a\cos\phi_p}\frac{\partial\cos\phi_p}{\partial\phi_{p,s}}$, $\frac{\partial}{\partial\tilde{x}_{p,s}}=\frac{\partial}{\partial x_{p,s}}=\frac{1}{a\cos\phi_p}\frac{\partial}{\partial\lambda_{p,s}}$, ∇_p and ∇_s are the horizontal gradients in a spherical coordinate system (with the above x and y coordinates, tilde is used when ∇ is used as curl or divergence), and $\mathbf{e}_\phi$ and $\mathbf{e}_r$ are the unit vectors in the latitudinal and vertical directions respectively.

Thermal wind balance

Using (A5) and (A2):

$$\frac{\partial}{\partial z}u^{(0)}=-\frac{1}{f}\frac{\partial\theta^{(2)}}{\partial y_p}, \tag{A9a}$$

$$\frac{\partial}{\partial z}\mathbf{u}^{(1)}=\frac{1}{f}\mathbf{e}_r\times\left(\nabla_p\theta^{(3)}+\nabla_s\theta^{(4)}\right). \tag{A9b}$$

Thermodynamic (θ) equations

Below is the list of all needed thermodynamic equations. Note that all orders below $\mathscr{O}(\varepsilon^5)$ give nothing, thus the first order that appears below is $\mathscr{O}(\varepsilon^5)$.

$\mathscr{O}(\varepsilon^5)$:

$$w^{(3)}=\frac{S_\theta^{(5)}}{\partial\theta^{(2)}/\partial z}=0 \tag{A10}$$

$\mathscr{O}(\varepsilon^6)$:

$$\frac{\partial\theta^{(3)}}{\partial t_p}+\frac{\partial\theta^{(4)}}{\partial t_s}+\frac{\partial}{\partial\tilde{x}_p}\left(u^{(0)}\theta^{(3)}\right)+\frac{\partial}{\partial\tilde{x}_s}\left(u^{(0)}\theta^{(4)}\right)+v^{(1)}\frac{\partial\theta^{(2)}}{\partial y_p}+w^{(4)}\frac{\partial\theta^{(2)}}{\partial z}=S_\theta^{(6)} \tag{A11}$$

$\mathscr{O}(\varepsilon^7)$:

$$\frac{\partial\theta^{(4)}}{\partial t_p}+\frac{\partial\theta^{(5)}}{\partial t_s}+\frac{\partial\theta^{(2)}}{\partial t_m}+\frac{\partial}{\partial\tilde{x}_p}\left(u^{(0)}\theta^{(4)}\right)+\mathbf{u}^{(1)}\cdot\nabla_p\theta^{(3)}+\mathbf{u}^{(1)}\cdot\nabla_s\theta^{(4)}$$
$$+\frac{\partial}{\partial\tilde{x}_s}\left(u^{(0)}\theta^{(5)}\right)+v^{(2)}\frac{\partial\theta^{(2)}}{\partial y_p}+w^{(4)}\frac{\partial\theta^{(3)}}{\partial z}+w^{(5)}\frac{\partial\theta^{(2)}}{\partial z}=S_\theta^{(7)} \tag{A12}$$

Continuity equations

This is the set of all continuity equations (also the trivial ones as they give us information about vertical velocities).

$\mathscr{O}(\varepsilon^0)$, $\mathscr{O}(\varepsilon^1)$ & $\mathscr{O}(\varepsilon^2)$:

$$\frac{\partial w^{(i)}}{\partial z}=0 \quad ; \quad i=0,1,2 \tag{A13}$$

$\mathscr{O}(\varepsilon^3)$ (here note that $w^{(3)}=0$ from the thermodynamic equation (A10) and that $\nabla_s\cdot\mathbf{u}^{(1)}=0$ by definition):

$$\nabla_p\cdot\mathbf{u}^{(0)}=0 \tag{A14}$$

$\mathscr{O}(\varepsilon^4)$:

$$\nabla_p\cdot\left(\mathbf{u}^{(1)}\rho^{(0)}\right)+\nabla_s\cdot\left(\mathbf{u}^{(2)}\rho^{(0)}\right)+\frac{\partial}{\partial z}\left(w^{(4)}\rho^{(0)}\right)=0 \tag{A15}$$

$\mathscr{O}(\varepsilon^5)$:

$$\nabla_p\cdot\left(\mathbf{u}^{(2)}\rho^{(0)}\right)+\nabla_s\cdot\left(\mathbf{u}^{(3)}\rho^{(0)}\right)+\frac{\partial}{\partial z}\left(w^{(5)}\rho^{(0)}\right)=0 \tag{A16}$$

$\mathscr{O}(\varepsilon^6)$:

$$\frac{\partial\rho^{(3)}}{\partial t_p}+\frac{\partial\rho^{(4)}}{\partial t_s}+\nabla_p\cdot\left(\mathbf{u}^{(3)}\rho^{(0)}+\mathbf{u}^{(1)}\rho^{(2)}+\mathbf{u}^{(0)}\rho^{(3)}\right)+$$
$$\nabla_s\cdot\left(\mathbf{u}^{(4)}\rho^{(0)}+\mathbf{u}^{(2)}\rho^{(2)}+\mathbf{u}^{(0)}\rho^{(4)}-\mathbf{u}^{(1)}\rho^{(0)}\frac{z}{a}\right)+\frac{\partial}{\partial z}\left(w^{(4)}\rho^{(2)}+w^{(6)}\rho^{(0)}\right)=0 \tag{A17}$$

$\mathscr{O}(\varepsilon^7)$:

$$\frac{\partial \rho^{(2)}}{\partial t_m}+\frac{\partial \rho^{(4)}}{\partial t_p}+\frac{\partial \rho^{(5)}}{\partial t_s}+\nabla_p\cdot\left(\mathbf{u}^{(4)}\rho^{(0)}+\mathbf{u}^{(2)}\rho^{(2)}+\mathbf{u}^{(1)}\rho^{(3)}+\mathbf{u}^{(0)}\rho^{(4)}-\mathbf{u}^{(1)}\rho^{(0)}\frac{z}{a}\right)$$
$$+\nabla_s\cdot\left(\mathbf{u}^{(5)}\rho^{(0)}+\mathbf{u}^{(3)}\rho^{(2)}+\mathbf{u}^{(2)}\rho^{(3)}+\mathbf{u}^{(1)}\rho^{(4)}+\mathbf{u}^{(0)}\rho^{(5)}-\mathbf{u}^{(2)}\rho^{(0)}\frac{z}{a}\right)$$
$$+\frac{\partial}{\partial z}\left(w^{(4)}\rho^{(3)}+w^{(5)}\rho^{(2)}+w^{(7)}\rho^{(0)}\right)=0 \quad \text{(A18)}$$

where terms with z/a come from corrections to the shallow-atmosphere approximation at higher orders. Note that these terms vanish in the zonal mean and/or synoptic scale average.

Vorticity Equation

To derive the vorticity equation, take $\nabla_s\times\mathscr{O}(\varepsilon^3)$ momentum equation (A6), and note that terms with $\nabla_s\times\nabla_s$ and synoptic scale derivatives of terms (π, ρ, θ) that do not depend on synoptic scales (up to 3^{rd} order) are zero. This yields (following DK13):

$$\frac{\partial}{\partial t_s}\boldsymbol{\zeta}^{(1)}+\nabla_s\times\left(\mathbf{u}^{(0)}\cdot\nabla_s\mathbf{u}^{(1)}\right)+\nabla_s\times\left(f\mathbf{e}_r\times\mathbf{u}^{(2)}\right)=-\nabla_s\times\nabla_p\pi^{(4)}+\nabla_s\times\mathbf{S}_u^{(3)} \quad \text{(A19)}$$

where $\nabla_s=\left((a\cos\phi_p)^{-1}\partial/\partial\lambda_s, a^{-1}\partial/\partial\phi_s\right)$, $\nabla_p=\left((a\cos\phi_p)^{-1}\partial/\partial\lambda_p, a^{-1}\partial/\partial\phi_p\right)$, the numbers in superscripts denote orders of variables, $\mathbf{u}=(u,v)$ is horizontal velocity, $\pi=p/\rho$, $\boldsymbol{\zeta}^{(1)}=\nabla_s\times\mathbf{u}^{(1)}$ is relative vorticity, and as ∇_s and $\mathbf{u}^{(1)}$ have only horizontal components $\boldsymbol{\zeta}^{(1)}=\zeta^{(1)}\mathbf{e}_r$. The source term $S_u^{(3)}$ represents frictional processes. Note that $\nabla_s\times\nabla_p\pi^{(4)}=(0,0,\nabla_p\cdot(f\mathbf{u}_s^{(1)}))$. Taking $\mathbf{e}_r\cdot$ of (A19) and applying the vector identities as in DK09 and DK13, we get:

$$\frac{\partial}{\partial t_s}\zeta^{(1)}+\mathbf{u}^{(0)}\cdot\nabla_s\zeta^{(1)}+f\nabla_s\cdot\mathbf{u}^{(2)}=-\nabla_p\cdot(f\mathbf{u}_s^{(1)})+\mathbf{e}_r\cdot\nabla_s\times\mathbf{S}_u^{(3)} \quad \text{(A20)}$$

where $S_\zeta=\mathbf{e}_r\cdot\nabla_s\times\mathbf{S}_u^{(3)}$ and $\nabla_p\cdot(f\mathbf{u}^{(1)})=f\nabla_p\cdot\mathbf{u}^{(1)}+v^{(1)}\cos\phi_p/a$ with $a^{-1}\cos\phi_p=a^{-1}\partial f/\partial\phi_p=\beta$. Since $\mathbf{u}^{(2)}$ is not known, we use the $\mathscr{O}(\varepsilon^4)$ continuity equation (A15) to obtain the vorticity equation:

$$\frac{\partial}{\partial t_s}\zeta^{(1)}+\mathbf{u}^{(0)}\cdot\nabla_s\zeta^{(1)}-\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\rho^{(0)}w^{(4)}\right)+\beta v^{(1)}=S_\zeta \quad \text{(A21)}$$

where $w^{(4)}$ is known from the $\mathscr{O}(\varepsilon^6)$ thermodynamic equation (A11), which can be used to derive the potential vorticity equation. This vorticity equation resembles the QG vorticity equation (e.g. Holton 2004), but now there are different scales represented in the equation.

APPENDIX B

Derivation of the Mean Flow Equations

a. Barotropic equation

This section shows the steps in deriving the barotropic pressure equation - combining the correct thermodynamic, hydrostatic, thermal wind, momentum and continuity equations (see Appendix A) with the $\mathscr{O}(\varepsilon^5)$ momentum equation (A8) averaged over t_s, λ_s, ϕ_s, t_p, λ_p, z (denoted with overline). Note that the vertical mean assumes $w = 0$ at the top and bottom boundaries. This section modifies the momentum (A8) and thermodynamic (A12) equations, which can then be used to derive the barotropic equations in section 4b (following DK13).

First average the flux forms of all equations mentioned:

Momentum Equations at $\mathscr{O}(\varepsilon^3)$, $\mathscr{O}(\varepsilon^4)$, $\mathscr{O}(\varepsilon^5)$:

$$\overline{v^{(2)}} = -\frac{\overline{S_u^{(3)}}^{s,p,z}}{f}, \tag{B1a}$$

$$\overline{v^{(3)}} = -\frac{\overline{S_u^{(4)}}^{s,p,z}}{f}, \tag{B1b}$$

$$\begin{aligned}\frac{\partial \overline{u^{(0)}\rho^{(0)}}^{s,p,z}}{\partial t_m} &+ \frac{\partial}{\partial \tilde{y}_p}\left(\overline{v^{(1)}u^{(1)}\rho^{(0)}}^{s,p,z} + \overline{v^{(2)}u^{(0)}\rho^{(0)}}^{s,p,z}\right) \\ -\frac{\tan\phi_p}{a}\left(\overline{v^{(1)}u^{(1)}\rho^{(0)}}^{s,p,z} + \overline{v^{(2)}u^{(0)}\rho^{(0)}}^{s,p,z}\right) &- \overline{\rho^{(0)}v^{(4)}f}^{s,p,z} \\ +\rho^{(0)}\overline{w^{(4)}}^{s,p,z}\cos\phi_p &= \overline{\rho^{(3)}\frac{\partial \pi^{(3)}}{\partial x_p}}^{s,p,z} + \overline{\rho^{(0)}S_u^{(5)}}^{s,p,z}.\end{aligned} \tag{B1c}$$

Continuity equations at $\mathscr{O}(\varepsilon^4)$, $\mathscr{O}(\varepsilon^5)$, $\mathscr{O}(\varepsilon^6)$, $\mathscr{O}(\varepsilon^7)$:

$$\frac{\partial}{\partial \tilde{y}_p}\left(\overline{v^{(1)}\rho^{(0)}}^{s,p,z}\right) = 0, \tag{B2a}$$

$$\frac{\partial}{\partial \tilde{y}_p}\left(\overline{v^{(2)}\rho^{(0)}}^{s,p,z}\right) = 0, \tag{B2b}$$

$$\frac{\partial}{\partial \tilde{y}_p}\left(\overline{v^{(3)}\rho^{(0)}}^{s,p,z}\right) = 0, \tag{B2c}$$

$$\frac{\partial \overline{\rho^{(2)}}^{s,p,z}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p}\left(\overline{v_p^{(1)}\rho^{(3)}}^{s,p,z} + \overline{v^{(2)}\rho^{(2)}}^{s,p,z} + \overline{v^{(4)}\rho^{(0)}}^{s,p,z}\right) = 0. \tag{B2d}$$

Thermodynamic equations at $\mathscr{O}(\varepsilon^6)$, $\mathscr{O}(\varepsilon^7)$:

$$\overline{w^{(4)}}^{s,p,z} = \frac{\overline{S_\theta^{(6)}}^{s,p,z}}{\partial \theta^{(2)}/\partial z}, \tag{B3a}$$

$$\frac{\partial \overline{\rho^{(0)}\theta^{(2)}}^{s,p,z}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p}\left(\overline{v_p^{(1)}\rho^{(0)}\theta^{(3)}}^{s,p,z} + \overline{v^{(2)}\rho^{(0)}\theta^{(2)}}^{s,p,z}\right) = \overline{S_\theta^{(7)}\rho^{(0)}}^{s,p,z}. \tag{B3b}$$

Hydrostatic balance at $\mathscr{O}(\varepsilon^2)$

$$\overline{\rho^{(2)}}^{s,p,z} = -\overline{\rho^{(0)}\theta^{(2)}}^{s,p,z} + \overline{p^{(2)}}^{s,p,z}. \tag{B4}$$

Equations (B1a,B1b) show that $\overline{v^{(2)}}^{s,p,z}$ and $\overline{v^{(3)}}^{s,p,z}$ are related to source terms, thus in the equations below they will be replaced by them. Note that $\rho^{(3)}\partial\pi^{(3)}/\partial x_p = f\rho^{(3)}v_p^{(1)}$ (via (A5)). Taking the hydrostatic balance equation (B4), using it to substitute $\rho^{(2)}$ in the continuity equation (B2d) and matching the $\partial\overline{\rho^{(0)}\theta^{(2)}}^{s,p,z}/\partial t_m$ term in the thermodynamic equation (B3b) yields

$$\frac{\partial \overline{p^{(2)}}^{s,p,z}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p}\left(\overline{v_p^{(1)}\rho^{(0)}\theta^{(3)}}^{s,p,z} + \overline{v_p^{(1)}\rho^{(3)}}^{s,p,z} + \overline{v^{(4)}\rho^{(0)}}^{s,p,z}\right)$$
$$= \overline{\rho^{(0)}S_\theta^{(7)}}^{s,p,z} + \frac{\partial}{\partial \tilde{y}_p}\left(\overline{\left(\rho^{(2)} + \rho^{(0)}\theta^{(2)}\right)\frac{S_u^{(3)}}{f}}^{s,p,z}\right). \tag{B5}$$

Rewriting the momentum equation then gives:

$$\frac{1}{f}\frac{\partial\overline{u^{(0)}\rho^{(0)}}^{s,p,z}}{\partial t_m}+\frac{1}{f}\frac{\partial}{\partial\tilde{y}_p}\left(\overline{v^{(1)}u^{(1)}\rho^{(0)}}^{s,p,z}\right)-\frac{1}{f}\frac{\tan\phi_p}{a}\left(\overline{v^{(1)}u^{(1)}\rho^{(0)}}^{s,p,z}\right)$$
$$-\overline{\rho^{(0)}v^{(4)}}^{s,p,z}-\overline{\rho^{(3)}v_p^{(1)}}^{s,p,z}=\frac{1}{f}\overline{\rho^{(0)}S_u^{(5)}}^{s,p,z}+\frac{1}{f}\frac{\partial}{\partial\tilde{y}_p}\left(\overline{\frac{S_u^{(3)}}{f}u^{(0)}\rho^{(0)}}^{s,p,z}\right)$$
$$-\frac{1}{f}\frac{\tan\phi_p}{a}\left(\overline{\frac{S_u^{(3)}}{f}u^{(0)}\rho^{(0)}}^{s,p,z}\right)-\frac{\rho^{(0)}\overline{S_\theta^{(6)}}^{s,p,z}\cos\phi_p}{f\partial\theta^{(2)}/\partial z}. \tag{B6}$$

The latter two equations are then used in section 4b to derive the barotropic pressure equation (16) or (17).

b. Baroclinic equation

This section shows the steps in deriving the baroclinic mean flow equation, which is derived through the $\mathscr{O}(\varepsilon^7)$ thermodynamic equation (A12) using the continuity and momentum equations averaged over t_s, λ_s, t_p, λ_p (denoted with an overbar). The averaged equations are:

Thermodynamic equations at $\mathscr{O}(\varepsilon^6)$, $\mathscr{O}(\varepsilon^7)$:

$$\overline{w^{(4)}}^{t_s,\lambda_s,p}=\frac{\overline{S_\theta^{(6)}}^{t_s,\lambda_s,p}}{\partial\theta^{(2)}/\partial z}, \tag{B7a}$$

$$\frac{\partial\overline{\rho^{(0)}\theta^{(2)}}^{t_s,\lambda_s,p}}{\partial t_m}+\frac{\partial}{\partial\tilde{y}_p}\left(\overline{v_p^{(1)}\rho^{(0)}\theta^{(3)}}^{t_s,\lambda_s,p}+\overline{v^{(2)}\rho^{(0)}\theta^{(2)}}^{t_s,\lambda_s,p}\right)$$
$$+\frac{\partial}{\partial\tilde{y}_s}\left(\overline{v_s^{(1)}\rho^{(0)}\theta^{(4)}}^{t_s,\lambda_s,p}+\overline{v^{(2)}\rho^{(0)}\theta^{(3)}}^{t_s,\lambda_s,p}-\overline{v^{(2)}\frac{z}{a}}^{t_s,\lambda_s,p}\right)$$
$$+\frac{\partial}{\partial z}\left(\overline{w^{(4)}\rho^{(0)}\theta^{(3)}}^{t_s,\lambda_s,p}-\overline{w^{(4)}\frac{z}{a}}^{t_s,\lambda_s,p}\right)+\overline{\rho^{(0)}w^{(5)}}^{t_s,\lambda_s,p}\frac{\partial\theta^{(2)}}{\partial z}=\overline{S_\theta^{(7)}\rho^{(0)}}^{t_s,\lambda_s,p}, \tag{B7b}$$

where terms with z/a come from corrections to the shallow-atmosphere approximation.

Continuity equations at $\mathscr{O}(\varepsilon^4)$, $\mathscr{O}(\varepsilon^5)$:

$$\frac{\partial}{\partial\tilde{y}_p}\left(\overline{v^{(1)}\rho^{(0)}}^{t_s,\lambda_s,p}\right)+\frac{\partial}{\partial\tilde{y}_s}\left(\overline{v^{(2)}\rho^{(0)}}^{t_s,\lambda_s,p}\right)+\frac{\partial}{\partial z}\left(\overline{w^{(4)}\rho^{(0)}}^{t_s,\lambda_s,p}\right)=0, \tag{B8a}$$

$$\frac{\partial}{\partial \tilde{y}_p}\left(\overline{v^{(2)}\rho^{(0)}}^{t_s,\lambda_s,p}\right)+\frac{\partial}{\partial \tilde{y}_s}\left(\overline{v^{(3)}\rho^{(0)}}^{t_s,\lambda_s,p}\right)+\frac{\partial}{\partial z}\left(\overline{w^{(5)}\rho^{(0)}}^{t_s,\lambda_s,p}\right)=0. \tag{B8b}$$

Momentum equations at $\mathscr{O}(\varepsilon^3)$, $\mathscr{O}(\varepsilon^4)$:

$$\overline{v^{(2)}}^{t_s,\lambda_s,p}=-\frac{\overline{S_u^{(3)}}^{t_s,\lambda_s,p}}{f}, \tag{B9a}$$

$$\overline{v^{(3)}}^{t_s,\lambda_s,p}=-\frac{\overline{S_u^{(4)}}^{t_s,\lambda_s,p}}{f}+\frac{\partial}{\partial \tilde{y}_s}\left(\frac{\overline{u_s^{(1)}v_s^{(1)}}^{t_s,\lambda_s,p}}{f}\right)+\frac{\overline{w^{(4)}}^{t_s,\lambda_s,p}}{f}\frac{\partial u^{(0)}}{\partial z}. \tag{B9b}$$

Here note that terms with $v_p^{(1)}\theta^{(3)}$ or $w^{(4)}\theta^{(3)}$, $v_p^{(1)}$ and $w^{(4)}$ cannot simply be averaged over λ_p and t_p - we need to average $v_p^{(1)}\theta^{(3)}$ or $w^{(4)}\theta^{(3)}$ together as also $\theta^{(3)}$ depends on planetary scales. This means that, in order to replace the $w^{(4)}$ and $v_p^{(1)}$ terms in equation (B7b), the $\mathscr{O}(\varepsilon^6)$ thermodynamic equation and $\mathscr{O}(\varepsilon^3)$ momentum equation have to first be multiplied by $\theta^{(3)}$ and then averaged over $\lambda_s, t_s, \lambda_p, t_p$. For the $\mathscr{O}(\varepsilon^3)$ momentum equation this gives

$$\overline{\theta^{(3)}v^{(2)}}^{t_s,\lambda_s,p}=-\frac{\overline{\theta^{(3)}S_u^{(3)}}^{t_s,\lambda_s,p}}{f}+\overline{\frac{\theta^{(3)}}{f}\frac{\partial \pi^{(4)}}{\partial x_p}}^{t_s,\lambda_s,p}. \tag{B10}$$

Multiplying equation (B10) by $\rho^{(0)}$ and taking $\partial/\partial\tilde{y}_s$ of it yields

$$\frac{\partial}{\partial \tilde{y}_s}\left(\overline{\rho^{(0)}\theta^{(3)}v^{(2)}}^{t_s,\lambda_s,p}\right)=-\frac{\partial}{\partial \tilde{y}_s}\left(\frac{\overline{\rho^{(0)}\theta^{(3)}S_u^{(3)}}^{t_s,\lambda_s,p}}{f}\right)+\overline{\rho^{(0)}u_s^{(1)}\frac{\partial \theta^{(3)}}{\partial x_p}}^{t_s,\lambda_s,p} \tag{B11}$$

where $u_s^{(1)}=-f^{-1}\partial\pi^{(4)}/\partial y_s$ was used. However, it is more complicated for the thermodynamic equation - here is a short derivation: First multiply the equation by $\theta^{(3)}$

$$\begin{aligned}\frac{1}{2}\frac{\partial \theta^{(3)^2}}{\partial t_p}+\frac{\partial \theta^{(3)}\theta^{(4)}}{\partial t_s}+\frac{1}{2}\frac{\partial}{\partial \tilde{x}_p}\left(u^{(0)}\theta^{(3)^2}\right)+\frac{\partial}{\partial \tilde{x}_s}\left(\theta^{(3)}u^{(0)}\theta^{(4)}\right)&\\ +\theta^{(3)}v^{(1)}\frac{\partial \theta^{(2)}}{\partial y_p}+\theta^{(3)}w^{(4)}\frac{\partial \theta^{(2)}}{\partial z}=\theta^{(3)}S_\theta^{(6)},&\end{aligned} \tag{B12}$$

then average it over $\lambda_s, t_s, \lambda_p, t_p$:

$$\overline{\theta^{(3)}w^{(4)}}^{t_s,\lambda_s,p}=-\overline{\theta^{(3)}v^{(1)}}^{t_s,\lambda_s,p}\frac{\partial\theta^{(2)}/\partial y_p}{\partial\theta^{(2)}/\partial z}+\frac{\overline{\theta^{(3)}S_\theta^{(6)}}^{t_s,\lambda_s,p}}{\partial\theta^{(2)}/\partial z}. \tag{B13}$$

We can derive an equation for $\overline{w^{(5)}\rho^{(0)}}^{t_s,\lambda_s,p}$ by integrating (B8b) over z and using (B9a) and (B9b). This yields:

$$\overline{w^{(5)}\rho^{(0)}}^{t_s,\lambda_s,p} = -\int_0^{z_{max}} \rho^{(0)} \frac{\partial}{\partial \tilde{y}_s}\left(\frac{\partial}{\partial \tilde{y}_s}\left(\frac{\overline{v_s^{(1)}u_s^{(1)}}^{t_s,\lambda_s,p}}{f}\right)\right) \mathrm{d}z + S_{w5} \qquad \text{(B14)}$$

with

$$S_{w5} = -\int_0^{z_{max}} \left[\frac{\partial}{\partial \tilde{y}_s}\left(\rho^{(0)}\left\{\frac{\overline{S_\theta^{(6)}}^{t_s,\lambda_s,p}}{f}\frac{\partial u^{(0)}/\partial z}{\partial \theta^{(2)}/\partial z} - \frac{\overline{S_u^{(4)}}^{t_s,\lambda_s,p}}{f}\right\}\right) - \frac{\partial}{\partial \tilde{y}_p}\left(\rho^{(0)}\frac{\overline{S_u^{(3)}}^{t_s,\lambda_s,p}}{f}\right)\right] \mathrm{d}z.$$

These equations are then used in section 4c to derive the final baroclinic equation for the mean flow (18, 19).

APPENDIX C

Derivation of the Angular Momentum Equation

This Appendix shows the derivation of angular momentum conservation for the zonal mean flow ($u^{(0)}$) equation, following from the $\mathscr{O}(\varepsilon^5)$ momentum equation (A8). Note that similar systems can be derived for higher order velocities as well and at all asymptotic orders, but are omitted for brevity.

Deriving an angular momentum equation for the mean flow means that something that corresponds to the fifth order momentum equation (A8) must be used. This means that, for example, $\mathrm{D}u/\mathrm{D}t$ has to be of fifth order, which overall makes the angular momentum equation (23) a second order equation, thus the rest of the terms in the equation must follow that pattern.

Using these statements and noting that $\phi = \phi_p$, the angular momentum equation (23) becomes

$$\varepsilon^{-3}\varepsilon^5 \frac{\mathrm{D}M}{\mathrm{D}t_m} = \varepsilon^{-3}\varepsilon^5 a\cos\phi_p \frac{\mathrm{D}u^{(0)}}{\mathrm{D}t_m} - (u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + ...)(\varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + ...)\sin\phi_p$$
$$-\varepsilon^{-2} f(v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + ...)a\cos\phi_p, \quad \text{(C1)}$$

where $v^{(0)} = 0$ because the zonal mean flow is geostrophic to leading order (A4). In this form, angular momentum is not conserved. To get a conservative form of this equation, multiply (C1) by $\rho = \rho^{(0)} + \varepsilon^2\rho^{(2)} + ...$

$$\varepsilon^2\rho\frac{\mathrm{D}M}{\mathrm{D}t_m} = \varepsilon^2 a\cos\phi_p(\rho^{(0)}+\varepsilon^2\rho^{(2)}+...)\frac{\mathrm{D}u^{(0)}}{\mathrm{D}t_m} - (\rho^{(0)}+\varepsilon^2\rho^{(2)}+...)(u^{(0)}+\varepsilon u^{(1)}+\varepsilon^2 u^{(2)}+...)(\varepsilon v^{(1)}+\varepsilon^2 v^{(2)}+...)\sin\phi_p - \varepsilon^{-2} f(\rho^{(0)}+\varepsilon^2\rho^{(2)}+...)(\varepsilon v^{(1)}+\varepsilon^2 v^{(2)}+...)a\cos\phi_p \tag{C2}$$

and taking the same orders together, yields the second order angular momentum equation (omit ε everywhere)

$$\rho\frac{\mathrm{D}M}{\mathrm{D}t_m} = a\cos\phi_p\rho^{(0)}\frac{\mathrm{D}u^{(0)}}{\mathrm{D}t_m} - (\rho^{(0)}u^{(1)}v^{(1)} + \rho^{(0)}u^{(0)}v^{(2)})\sin\phi_p - f(\rho^{(0)}v^{(4)} + \rho^{(2)}v^{(2)} + \rho^{(3)}v^{(1)})a\cos\phi_p. \tag{C3}$$

Note that since an equation for the mean flow is derived, (24) can be averaged over synoptic scales (t_s, λ_s, ϕ_s) and planetary time scale (t_p), which simplifies it. To get the angular conservation equation, the continuity equations (A14-A16) are needed, which can be written together as

$$\nabla_p\cdot(\overline{\rho^{(0)}\mathbf{u}^{(i)}}^{s,t_p}) + \frac{\partial(\overline{\rho^{(0)}w^{(i+3)})}^{s,t_p}}{\partial z} = 0 \tag{C4}$$

where overline denotes average over $(t_s, t_p, \lambda_s, \phi_s)$, and $i = 0, 1, 2$ (where for $i = 0$, $w^{(3)} = 0$). This equation can then be written in a shorter form as

$$\nabla_p^{3D}\cdot(\overline{\rho^{(0)}\mathbf{u}_{3D}^{(i)}}^{s,t_p}) = 0 \tag{C5}$$

where

$$\nabla_p^{3D}\cdot = \left(\frac{1}{a\cos\phi_p}\frac{\partial}{\partial\lambda_p}, \frac{1}{a\cos\phi_p}\frac{\partial\cos\phi_p}{\partial\phi_p}, \frac{\partial}{\partial z}\right)$$

now includes the vertical derivative and $\mathbf{u}_{3D}^{(i)} = (u^{(i)}, v^{(i)}, w^{(i+3)})$ is the three-dimensional velocity field. Note that in general the continuity equation can be used to simplify expression (24), using

$$\rho\frac{\mathrm{D}B}{\mathrm{D}t} = \frac{\mathrm{D}\rho B}{\mathrm{D}t} - B\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial(\rho B)}{\partial t} + \nabla^{3D}\cdot(B\rho\mathbf{u}_{3D}) \tag{C6}$$

where B is an arbitrary scalar, and $\mathbf{u}_{3D}$ is three-dimensional velocity; noting that mass is conserved for every order, the continuity equation for each order in general takes the form $\mathrm{D}\rho/\mathrm{D}t = -\rho\nabla_{3D}\cdot\mathbf{u}$, where $\partial\rho/\partial t$ is mainly zero as $\rho^{(0)}$ only depends on the vertical coordinate.

Using (C6) for $\rho\mathrm{D}M/\mathrm{D}t_m$ and (C5, A8) for $\rho^{(0)}\mathrm{D}u^{(0)}/\mathrm{D}t_m$ gives

$$\frac{\partial\overline{(\rho M)}^{s,t_p}}{\partial t_m} + \nabla_p^{3D}\cdot\overline{(M\rho\mathbf{u}_{3D})}^{s,t_p} = a\cos\phi_p\frac{\partial\overline{(\rho^{(0)}u^{(0)})}^{s,t_p}}{\partial t_m}$$
$$+a\cos\phi_p\nabla_p^{3D}\cdot\left(\overline{u^{(2)}\rho^{(0)}\mathbf{u}_{3D}^{(0)}}^{s,t_p} + \overline{u^{(1)}\rho^{(0)}\mathbf{u}_{3D}^{(1)}}^{s,t_p} + \overline{u^{(0)}\rho^{(0)}\mathbf{u}_{3D}^{(2)}}^{s,t_p}\right)$$
$$-(\overline{\rho^{(0)}u^{(1)}v^{(1)}}^{s,t_p} + \overline{\rho^{(0)}u^{(0)}v^{(2)}}^{s,t_p})\sin\phi_p - f(\overline{\rho^{(0)}v^{(4)}}^{s,t_p} + \overline{\rho^{(2)}v^{(2)}}^{s,t_p} + \overline{\rho^{(3)}v^{(1)}}^{s,t_p})a\cos\phi_p. \tag{C7}$$

Note that the orders of separate terms on the right hand side are not given as they do not play an important role in the further derivation (for simplicity), however note that overall $\overline{\rho M}^{s,t_p}$ and $\overline{M\rho\mathbf{u}_{3D}}^{s,t_p}$ are of the second order.

From (A8) multiplied by $\rho^{(0)}$ it follows that

$$\overline{\rho^{(0)}\frac{Du^{(0)}}{Dt_m}}^{s,t_p} = f(\overline{v^{(4)}\rho^{(0)}}^{s,t_p} + \overline{v^{(1)}\rho^{(3)}}^{s,t_p} + \overline{v^{(2)}\rho^{(2)}}^{s,t_p})$$
$$+\frac{\tan\phi_p}{a}\left(\overline{v^{(2)}u^{(0)}\rho^{(0)}}^{s,t_p} + \overline{v^{(1)}u^{(1)}\rho^{(0)}}^{s,t_p}\right) + \overline{\rho^{(0)}S_u^{(5)}}^{s,t_p} - \frac{\partial}{\partial x_p}\left(\overline{\pi^{(6)}\rho^{(0)}}^{s,t_p}\right)$$
$$-\overline{\frac{\cos\phi_p}{\partial\theta^{(2)}/\partial z}S_\theta^{(6)}}^{s,t_p} + \overline{\rho^{(2)}S_u^{(3)}}^{s,t_p} + \frac{\partial}{\partial x_p}\left[\frac{\cos\phi_p}{\partial\theta^{(2)}/\partial z}\left(\overline{u^{(0)}\theta^{(3)}\rho^{(0)}}^{s,t_p} + \overline{\frac{\rho^{(0)}\pi^{(3)}}{f}\frac{\partial\theta^{(2)}}{\partial y_p}}^{s,t_p}\right)\right] \tag{C8}$$

where the last two terms come from the $w^{(4)}\cos\phi_p$ term using the thermodynamic equation (A11) averaged over synoptic scales and t_p, $fv^{(1)}\rho^{(3)} = \rho^{(3)}\partial\pi^{(3)}/\partial x_p$ (via (A5)), and $\overline{fv^{(2)}\rho^{(2)}}^{s,t_p} = \overline{\pi^{(4)}\rho^{(2)}}^{s,t_p} + \overline{\rho^{(2)}S_u^{(3)}}^{s,t_p}$ (via (A6)). Notice that the first two terms on the right-hand-side of (C8) resemble the terms involving $\sin\phi_p$ and $fa\cos\phi_p$ in (C7), and lead to a cancellation after combining (C7) and (C8). The terms that remain in the equation can all be integrated over a volume V_p (λ_p, ϕ_p, z). Following Gauss' theorem[5], assuming no source-sink terms and assuming there is no orography (for simplicity) yields angular momentum conservation

$$\iiint_{V_p} \frac{\partial\overline{(\rho M)}^{s,t_p}}{\partial t_m} \mathrm{d}V_p = 0. \tag{C9}$$

The angular momentum equation can be linked to the barotropic pressure equation (17) using (C7), dividing it first by $a\cos\phi_p$, then integrating it over a longitude-height slice (over area A_p, which effectively gives additional averaging over λ_p and z) and using the divergence theorem again which gives

$$\frac{1}{a\cos\phi_p}\left[\frac{\partial\overline{(\rho M)}^{s,p,z}}{\partial t_m} + \frac{\partial}{\partial\tilde{y}_p}\overline{(M\rho v)}^{s,p,z}\right] = \frac{\partial\overline{\rho^{(0)}u^{(0)}}^{s,p,z}}{\partial t_m}$$
$$+\frac{\partial}{\partial\tilde{y}_p}\left(\overline{u^{(1)}\rho^{(0)}v^{(1)}}^{s,p,z} + \overline{u^{(0)}\rho^{(0)}v^{(2)}}^{s,p,z}\right) - \left(\overline{\rho^{(0)}u^{(1)}v^{(1)}}^{s,p,z} + \overline{\rho^{(0)}u^{(0)}v^{(2)}}^{s,p,z}\right)\frac{\tan\phi_p}{a}$$
$$-f\left(\overline{\rho^{(0)}v^{(4)}}^{s,p,z} + \overline{\rho^{(2)}v^{(2)}}^{s,p,z} + \overline{\rho^{(3)}v^{(1)}}^{s,p,z}\right). \tag{C10}$$

Here the overbar denotes an average over $t_s, t_p, \lambda_s, \lambda_p, \phi_s, z$ and note that $v^{(2)}$ is proportional to a source term under such an average (B1a). Now divide (C10) by f, take $\partial/\partial\tilde{y}_p$ of it, and finally

[5]Gauss' theorem generally states $\iiint_V \nabla\cdot\mathbf{F}\mathrm{d}V = \iint_{\partial V}\mathbf{F}\cdot\mathbf{n}\mathrm{d}S$, where $\mathbf{F}$ is a three-dimensional vector, $\mathbf{n}$ is a normal vector on surface S, and ∂V is the surface around the volume V of interest. Note that in the case of $\mathbf{F} = \rho M\mathbf{u}$ the $\iint_{\partial V}\mathbf{F}\cdot\mathbf{n}\mathrm{d}S = 0$ as $\mathbf{u}\cdot\mathbf{n} = 0$ at the lower boundary and $\rho\to 0$ at the upper boundary.

multiply it by f. This yields

$$\mathscr{L}\left\{\frac{1}{a\cos\phi_p}\left[\frac{\partial\overline{\rho M}^{s,p,z}}{\partial t_m}+\frac{\partial}{\partial\tilde{y}_p}\left(\overline{M\rho v}^{s,p,z}\right)\right]\right\}=\mathscr{L}\left\{\frac{\partial\overline{\rho^{(0)}u^{(0)}}^{s,p,z}}{\partial t_m}\right\}$$
$$+\mathscr{L}\left\{\frac{\partial}{\partial\tilde{y}_p}\left(\overline{u^{(1)}\rho^{(0)}v^{(1)}}^{s,p,z}\right)-(\overline{\rho^{(0)}u^{(1)}v^{(1)}}^{s,p,z})\frac{\tan\phi_p}{a}\right\}$$
$$-f\frac{\partial}{\partial\tilde{y}_p}\left(\overline{\rho^{(0)}v^{(4)}}^{s,p,z}+\overline{\rho^{(2)}v^{(2)}}^{s,p,z}+\overline{\rho^{(3)}v^{(1)}}^{s,p,z}\right), \tag{C11}$$

where source terms were omitted for simplicity, the left-hand-side can be simplified to

$$\mathscr{L}\left\{\overline{\frac{\rho}{a\cos\phi_p}\frac{DM}{Dt_m}}^{s,p,z}\right\}$$

with

$$\mathscr{L}=\frac{\partial}{\partial\tilde{y}_p}-\frac{\beta}{f},$$

and the last term in the equation can be simplified to $+f\partial\rho^{(2)}/\partial t_m$ via (B2d). Notice how all but the last term on the right-hand-side resemble terms in the barotropic pressure equation (17). This means that (17) can be rewritten using the angular momentum equation as

$$\mathscr{L}\left\{\overline{\frac{\rho}{a\cos\phi_p}\frac{DM}{Dt_m}}^{s,p,z}\right\}-f\frac{\partial\overline{\rho^{(2)}}^{s,p,z}}{\partial t_m}=-f\frac{\partial\overline{p^{(2)}}^{s,p,z}}{\partial t_m}-f\frac{\partial}{\partial\tilde{y}_p}\left(\overline{\rho^{(0)}v_p^{(1)}\theta^{(3)}}^{s,p,z}\right) \tag{C12}$$

where $\rho^{(2)}=p^{(2)}-\rho^{(0)}\theta^{(2)}$ via (B4), which further simplifies it. This now gives a clear link between the barotropic equation for the mean flow and the angular momentum.

APPENDIX D

The Non-acceleration Theorem

This Appendix shows the derivation of the non-acceleration theorem for the given asymptotic set of equations. To derive this, a Transformed Eulerian Mean (TEM) (Andrews and McIntyre 1976; Edmon et al. 1980) version of the zonal mean (averaged over λ_p,λ_s, denoted by [.]) momentum and thermodynamic equations is necessary. From the zonal mean continuity ($\mathscr{O}(\varepsilon^4,\varepsilon^5)$), thermodynamic ($\mathscr{O}(\varepsilon^6,\varepsilon^7)$) and momentum equations ($\mathscr{O}(\varepsilon^3,\varepsilon^4,\varepsilon^5)$) at different asymptotic orders, we can identify the residual meridional circulation ($v_r^{(i)}$, $w_r^{(i)}$ with subscript r representing residual velocity and i represents its order)

$$[\rho^{(0)}v_r^{(2)}] = [\rho^{(0)}v^{(2)}] - \frac{\partial}{\partial z}\left[\frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\right] \tag{D1}$$

$$[\rho^{(0)}w_r^{(4)}] = [\rho^{(0)}w^{(4)}] + \frac{\partial}{\partial\tilde{y}_s}\left[\frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\right] = [\rho^{(0)}w^{(4)}] \tag{D2}$$

$$[\rho^{(0)}v_r^{(3)}] = [\rho^{(0)}v^{(3)}] - \frac{\partial}{\partial z}\left[\frac{v_s^{(1)}\theta^{(4)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\right] \tag{D3}$$

$$[\rho^{(0)}w_r^{(5)}] = [\rho^{(0)}w^{(5)}] + \frac{\partial}{\partial\tilde{y}_p}\left[\frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\right] + \frac{\partial}{\partial\tilde{y}_s}\left[\frac{v_s^{(1)}\theta^{(4)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\right], \tag{D4}$$

which satisfy continuity equations at different orders.

Using the residual velocities (D1-D4), the zonal mean momentum equations at $\mathscr{O}(\varepsilon^3,\varepsilon^4)$ (A6, A7) become

$$\frac{\partial[\rho^{(0)}u^{(1)}]}{\partial t_s} - f[\rho^{(0)}v_r^{(2)}] = [\rho^{(0)}S_u^{(3)}] + \frac{\partial}{\partial z}\left[\frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\right], \tag{D5}$$

$$\begin{aligned}\frac{\partial[\rho^{(0)}u^{(2)}]}{\partial t_s} + \frac{\partial[\rho^{(0)}u^{(1)}]}{\partial t_p} + [\rho^{(0)}w_r^{(4)}]\frac{\partial u^{(0)}}{\partial z} - f[\rho^{(0)}v_r^{(3)}] \\ = [\rho^{(0)}S_u^{(4)}] - \frac{\partial}{\partial\tilde{y}_s}\left[\rho^{(0)}u_s^{(1)}v_s^{(1)}\right] + \frac{\partial}{\partial z}\left[\frac{v_s^{(1)}\theta^{(4)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z}\right],\end{aligned} \tag{D6}$$

which can both be linked to the zonal mean wave activity equations on planetary (13) and synoptic (12) scales, respectively, through their respective zonal mean EP flux divergences ($[\nabla_p^{3D}\cdot\mathbf{F}_p]$, $[\nabla_s^{3D}\cdot\mathbf{F}_s]$) that appear on the right-hand-side of (D5, D6). Thus, (D5, D6) can be rewritten in terms of wave activities as

$$\frac{\partial[\rho^{(0)}u^{(1)}]}{\partial t_s} + \frac{\partial[\mathscr{A}_p]}{\partial t_p} = f[\rho^{(0)}v_r^{(2)}] + [\rho^{(0)}S_u^{(3)}] + [S_p^{wa}], \tag{D7}$$

$$\frac{\partial[\rho^{(0)}u^{(2)}]}{\partial t_s} + \frac{\partial[\rho^{(0)}u^{(1)}]}{\partial t_p} + \frac{\partial[\mathscr{A}_s]}{\partial t_s} = f[\rho^{(0)}v_r^{(3)}] - [\rho^{(0)}w_r^{(4)}]\frac{\partial u^{(0)}}{\partial z} + [\rho^{(0)}S_u^{(4)}] + [S_s^{wa}], \tag{D8}$$

which, under synoptic scale averaging (ϕ_s, t_s), for steady eddies (wave activity tendencies vanish), and in the absence of source-sink terms, satisfy the non-acceleration theorem, i.e. the tendencies of the zonal mean velocities vanish. These equations also show that planetary wave activity affects the zonal mean flow evolution on synoptic timescales, and that the synoptic wave activity (linked to synoptic heat and momentum fluxes) affects the zonal mean flow evolution on planetary timescales. However, the latter relationship vanishes under synoptic scale averaging, leaving only the residual circulation terms and source-sink terms affecting the evolution of $u_p^{(1)}$ in (D8). This means that an evolution equation for $p^{(3)}$ (related to $u_p^{(1)}$), which can be derived in a similar manner as the barotropic equation (evolution equation for $p^{(2)}$) using $\mathscr{O}(\varepsilon^4)$ u-momentum equation, $\mathscr{O}(\varepsilon^6)$ thermodynamic equation, $\mathscr{O}(\varepsilon^6)$ continuity equation, and hydrostatic balance for $p^{(3)}$ averaged over synoptic scales and vertically, is only affected by the source-sink terms

$$\left(\frac{\partial}{\partial \tilde{y}_p}\frac{1}{f}\frac{\partial}{\partial y_p} - \frac{\beta}{f^2}\frac{\partial}{\partial y_p} - f\right)\frac{\partial \overline{p^{(3)}}^{\lambda_p,s,z}}{\partial t_p} = -\overline{f\rho^{(0)}S_\theta^{(6)}}^{\lambda_p,s,z} - \left(\frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f}\right)\left(\overline{\rho^{(0)}S_u^{(4)}}^{\lambda_p,s,z}\right). \quad \text{(D9)}$$

This evolution equation suggests that a higher order momentum equation is needed to find the dynamic influences on the mean flow on planetary spatial scales (averaged over synoptic scales) and longer time scales (t_m) - see barotropic pressure equation (16).

Note that (D7,D8) provide equations for zonal mean flow variations on shorter timescales (synoptic and planetary), which have dynamical importance for higher frequency atmospheric flow (e.g. baroclinic life cycles or barotropic annular modes with timescales of 10 days or less). Upon averaging over these scales, the slower variations in the mean flow (t_m) emerge (as in the barotropic equation for the mean flow).

The TEM version of the $\mathscr{O}(\varepsilon^5)$ zonal momentum equation can also be derived using the same residual velocities (with the same procedure), however, here we only show an equation averaged over $t_s, t_p, \lambda_s, \lambda_p, \phi_s, z$ as this was the averaging performed to derive the barotropic equation for the mean flow (17). This yields

$$\frac{\partial \overline{\rho^{(0)}u^{(0)}}^{p,s,z}}{\partial t_m} + \overline{\rho^{(0)}v_r^{(2)}\frac{\partial u^{(0)}}{\partial \tilde{y}_p}}^{p,s,z} + \overline{\rho^{(0)}w_r^{(5)}\frac{\partial u^{(0)}}{\partial z}}^{p,s,z} + \overline{\rho^{(0)}w_r^{(4)}}^{p,s,z}\cos\phi_p$$
$$- f\overline{\rho^{(0)}v^{(4)}}^{p,s,z} - \overline{f\rho^{(3)}v_p^{(1)}}^{p,s,z} = \overline{\rho^{(0)}S_u^{(5)}}^{p,s,z} + \frac{\partial \overline{F^y}^{p,s,z}}{\partial \tilde{y}_p} \tag{D10}$$

with

$$F^y = -\rho^{(0)}u^{(1)}v^{(1)}\cos\phi_p + \frac{\partial u^{(0)}}{\partial z}\frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \tag{D11}$$

where $a^{-1}\tan\phi_p\overline{\rho^{(0)}u^{(1)}v^{(1)}}^{p,s,z}$ was absorbed into F^y through $\cos\phi_p$. As in section 4b, many terms in (D10) can be related to source-sink terms, $v^{(4)}$ can be eliminated via the continuity and thermodynamic equations, and $f\rho^{(3)}v_p^{(1)}$ is related to meridional heat flux on planetary scales. To link (D10) to the wave activity tendency, a higher order wave activity approximation would be needed, and due to the planetary scale heat fluxes in (D10), also a boundary wave activity may be needed, which are not the subject of this paper (only the leading order approximations are of interest). Hence a non-acceleration theorem for this order of the momentum equation is yet to be determined, but is expected to hold as is the case at lower orders.

The $\mathscr{O}(\varepsilon^7)$ thermodynamic equation within the TEM framework (under a $t_s, t_p, \lambda_s, \lambda_p, \phi_s$ average) is

$$\frac{\partial \overline{\rho^{(0)}\theta^{(2)}}^{s,p}}{\partial t_m} + \overline{\rho^{(0)}v_r^{(2)}}^{s,p}\frac{\partial\theta^{(2)}}{\partial y_p} + \overline{\rho^{(0)}w_r^{(5)}}^{s,p}\frac{\partial\theta^{(2)}}{\partial z} = \overline{\rho^{(0)}S_\theta^{(7)}}^{s,p} - \frac{\partial}{\partial z}\left(\frac{\overline{S_\theta^{(6)}\theta^{(3)}\rho^{(0)}}^{s,p}}{\partial\theta^{(2)}/\partial z}\right), \tag{D12}$$

which completes the TEM version of the equations. Note that the $\mathscr{O}(\varepsilon^6)$ thermodynamic equation remains unchanged within the TEM framework and is hence not repeated here.

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LIST OF TABLES

TABLE 1. The assumptions for the scale separations between planetary (p), synoptic (s) and zonal mean flow (m).

	longitude	latitude	height	time
planetary	$\lambda_p = \lambda$	$\phi_p = \phi$	$z_p = z_s = z$	$t_p = \varepsilon^3 t$
synoptic	$\lambda_s = \varepsilon^{-1}\lambda_p$	$\phi_s = \varepsilon^{-1}\phi_p$	$z_p = z_s = z$	$t_s = \varepsilon^2 t = \varepsilon^{-1} t_p$
mean		$\phi_m = \phi_p$	$z_m = z_p = z$	$t_m = \varepsilon^5 t = \varepsilon^2 t_p$