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potential/Conservative Temperature and
implications for ocean modelling*

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Accepted Version

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Tailleux, R. ORCID: <https://orcid.org/0000-0001-8998-9107>
(2015) Observational and energetics constraints on the non-
conservation of potential/Conservative Temperature and
implications for ocean modelling. *Ocean Modelling*, 88. pp. 26-
37. ISSN 1463-5003 doi: 10.1016/j.ocemod.2015.02.001
Available at <https://centaur.reading.ac.uk/39239/>

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To link to this article DOI: <http://dx.doi.org/10.1016/j.ocemod.2015.02.001>

Publisher: Elsevier

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Observational and energetics constraints on the non-conservation of potential/Conservative Temperature and implications for ocean modelling

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Abstract

This paper seeks to elucidate the fundamental differences between the non-conservation of potential temperature and that of Conservative Temperature, in order to better understand the relative merits of each quantity for use as the heat variable in numerical ocean models. The main result is that potential temperature is found to behave similarly to entropy, in the sense that its nonconservation primarily reflects production/destruction by surface heat and freshwater fluxes; in contrast, the nonconservation of Conservative Temperature is found to reflect primarily the overall compressible work of expansion/contraction. This paper then shows how this can be exploited to constrain the nonconservation of potential temperature and entropy from observed surface heat fluxes, and the nonconservation of Conservative Temperature from published estimates of the mechanical energy budgets of ocean numerical models. Finally, the paper shows how to modify the evolution equation for potential temperature so that it is exactly equivalent to using an exactly conservative evolution equation for Conservative Temperature, as was recently recommended by IOC et al. (2010). This result should in

principle allow ocean modellers to test the equivalence between the two formulations, and to indirectly investigate to what extent the budget of derived nonconservative quantities such as buoyancy and entropy can be expected to be accurately represented in ocean models.

Keywords:

ocean modelling, conservation equations, heat non-conservation, energy conservation, potential temperature, Conservative Temperature

1. Introduction

The issue of whether potential temperature θ is the most appropriate heat variable to be used in numerical ocean general circulation models (OGCMs) has recently come under scrutiny following McDougall (2003) and IOC et al. (2010) suggestion that θ should be replaced by Conservative Temperature (CT or Θ). The main argument, originally made by McDougall (2003), is that the current practice of treating θ as a conservative quantity is significantly inaccurate, and to the extent that one should insist in treating heat as conservative in OGCMs, it appears to be significantly more accurate to do so by using CT instead. Since CT is a relatively new quantity, its formal properties have yet to be fully understood, so that the full implications of switching from θ to CT in OGCMs are not all entirely clear. The alternative option — to accept that potential temperature is fundamentally nonconservative and to modify its model formulation accordingly — has not received attention so far, but needs to be understood to inform the debate about whether to switch or not. The main purpose of this paper is to achieve a deeper understanding of the fundamental differences between the nonconser-

18 vation of potential temperature and that of Conservative Temperature, in
19 order to help ocean modellers better understand the pros and cons of each
20 modelling choice.

21 From a fundamental viewpoint, the nonconservation of heat (arising from
22 irreversible processes) as measured by potential temperature, Conservative
23 Temperature or entropy is now well understood to be a natural consequence of
24 energy conservation, in the sense that were heat to be conservative, total en-
25 ergy would be nonconservative and conversely, as shown in Tailleux (2010a).
26 Tailleux (2010a) proposed to extend this idea to coarse-grained primitive hy-
27 drostatic Boussinesq models as a practical way (and somewhat ad-hoc) to
28 evaluating the nonconservation of θ and CT in such models. Specifically, the
29 method works as follows. Starting from the hydrostatic Boussinesq primitive
30 equations formulated in terms of either potential temperature, Conservative
31 Temperature or entropy, one writes down the evolution equation for the total
32 energy assuming that the evolution equation for any of the heat variables is
33 the sum of a conservative part (expressed as the divergence of some flux)
34 plus an a priori unknown nonconservative part. Both conservative and non-
35 conservative terms appear in the equation for total energy thus obtained. As
36 discussed by Tailleux (2010a), each nonconservative term is associated with
37 some inconsistency in the model formulation, such as using the total hydro-
38 static pressure instead of the Boussinesq pressure in the equation of state
39 for instance. One of the nonconservative terms in the total energy equation
40 is directly related to the heat nonconservation term. Imposing such a term
41 to be zero, as required by the principle of energy conservation, provides an
42 explicit expression for the heat nonconservative term in terms of the heat

43 and salt fluxes, which depends on the assumed form of the turbulent ocean
 44 mixing processes. Graham and McDougall (2013) uses a similar approach
 45 to quantify heat nonconservation, but which relies on the existence of a dif-
 46 ferent conservative quantity (a locally referenced potential enthalpy) than
 47 total energy. Their approach yields a different expression for the noncon-
 48 servative terms than that of Tailleux (2010a), which among other things,
 49 lacks pressure gradient terms. Although their approaches rely on different
 50 assumptions, it is important to point out that Tailleux (2010a) and Graham
 51 and McDougall (2013) nevertheless agree that the expression for the non-
 52 conservation of θ and Θ in terms of the turbulent fluxes follows directly from
 53 the particular quantity that one assumes to be conservative for the averaged
 54 equations of motion. As an alternative to Tailleux (2010a), who assumed
 55 the latter to be total energy, and to Graham and McDougall (2013), who
 56 assumed it to be a locally defined form of potential enthalpy, one may sim-
 57 ply assume CT to be *exactly* conservative, as recommended by IOC et al.
 58 (2010), and derive the implied form for the nonconservation of θ . Such an
 59 idea will be exploited in Section 5.

60 The main objective of this paper is to clarify the nature of the non-
 61 conservation of potential/Conservative Temperature and of entropy. Building
 62 upon the results by Tailleux (2012), we argue that the nonconservative
 63 production of Conservative Temperature or potential enthalpy fundamentally
 64 measures the thermodynamic work of expansion/contraction. In other words,
 65 in the same way that McDougall (2003) argues that potential enthalpy is
 66 the most appropriate variable to measure “heat” into the oceans, we argue
 67 that the non-conservation of potential enthalpy measures the “work” done

by compressible effects. We also argue that the nonconservative production of potential temperature is of a fundamentally different nature, and actually measures the production of potential temperature by surface heat fluxes (and to a lesser extent freshwater fluxes), in the same way that irreversible entropy production reflects the entropy production by surface fluxes in a steady-state system.

This paper is organised as follows. Section 2 recalls the general construction of the nonconservative production terms and their link to energy conservation initiated in Tailleux (2010a) and Tailleux (2012), and further shows how to link the non conservation of potential temperature and entropy to ocean surface properties; such a link is well known for entropy, but not for potential temperature. Section 3 discusses a priori estimates for the non conservation terms, as well as some of their theoretical properties. Section 4 uses observations to illustrate and quantify empirically the results of Section 3. Section 5 offers a summary and discussion of the implications of our results for ocean modelling, which leads us to propose a modification of the evolution equation for θ that is meant to be equivalent to a strictly conservative evolution equation for Θ and hence that we propose as a basis for informing the debate about switching or not.

2. Non-conservation of “heat” variables for the fully compressible Navier-Stokes equations for seawater

2.1. Compressible Navier-Stokes equations and “heat” variables

The nonconservativeness of different measures of heat is discussed in the context of the full compressible Navier-Stokes equations (NCS) in a rotating

92 frame, viz.,

$$\rho \frac{D\mathbf{v}}{Dt} + 2\mathbf{\Omega} \times (\rho\mathbf{v}) + \nabla P = -\rho\nabla\Phi + \rho\nabla \cdot \mathbf{S} \quad (1)$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0 \quad (2)$$

$$\rho \frac{DS}{Dt} = -\nabla \cdot (\rho\mathbf{F}_S) \quad (3)$$

95 where $\mathbf{v} = (u, v, w)$ is the three-dimensional velocity field, $D/Dt = \partial_t + \mathbf{v} \cdot \nabla$
 96 is [the](#) substantial derivative, P is the pressure, ρ is the density, $\Phi = g_0 Z$ is the
 97 geopotential, g_0 is the acceleration of gravity, $\mathbf{\Omega}$ is Earth's rotation vector,
 98 $\mathbf{S}$ is the stress tensor, $\Phi = g_0 Z$ is the geopotential formulated in terms
 99 of a constant gravitational potential acceleration g_0 and [geometric](#) height
 100 $Z = z$, with z the regular height increasing upwards. [Chemical composition](#)
 101 [is described by the salinity \$S\$ \(which in practice one may assume to be the](#)
 102 [Absolute Salinity defined in IOC et al. \(2010\)\), and \$\mathbf{F}_S\$ is the diffusive](#)
 103 [salinity flux.](#)

104 2.2. Heat variables

105 As in Graham and McDougall (2013), we focus on the classical measures
 106 of heat based on specific entropy η and potential temperature θ , as well as
 107 on the more recent Conservative Temperature Θ . As these variables are
 108 all nonconservative, they can all *a priori* be assumed to satisfy evolution
 109 equations of the form

$$\rho \frac{D\eta}{Dt} = -\nabla \cdot (\rho\mathbf{F}_\eta) + \rho\dot{\eta}_{irr}, \quad (4)$$

$$\rho \frac{D\theta}{Dt} = -\nabla \cdot (\rho\mathbf{F}_\theta) + \rho\dot{\theta}_{irr}, \quad (5)$$

$$\rho \frac{D\Theta}{Dt} = -\nabla \cdot (\rho\mathbf{F}_\Theta) + \rho\dot{\Theta}_{irr}, \quad (6)$$

where $\mathbf{F}_\eta$, $\mathbf{F}_\Theta$, and $\mathbf{F}_\theta$ are the fluxes of each quantity, involving radiative effects, molecular diffusion, and latent heat release, while $\dot{\eta}_{irr}$, $\dot{\theta}_{irr}$ and $\dot{\Theta}_{irr}$ are the non-conservation terms for each quantity that are the main focus of the present paper.

Physically, potential temperature θ represents the temperature that a parcel would have if brought [to the surface adiabatically at constant composition](#), and is therefore fundamentally linked to entropy, being implicitly defined by the relation

$$\eta(\theta, S, P_0) = \eta(T, S, P) \quad (7)$$

where P_0 is a reference mean atmospheric pressure, and T the in-situ temperature. Note that throughout the manuscript, both θ and T denote absolute temperatures expressed in [kelvin](#), as some of the quantities discussed below, such as the ratio T/θ would not make sense if T and θ were expressed [on the Celsius scale](#). In contrast, Conservative Temperature is defined as being proportional to potential enthalpy h_θ , such that $c_p^0 \Theta = h_\theta$, with the potential enthalpy being defined as the enthalpy that a parcel would have if brought adiabatically to the surface, and thus implicitly defined by the relation

$$h_\theta = h(\eta, S, P_0), \quad (8)$$

[or equivalently as \$\eta\(h, S, P\) = \eta\(h_\theta, S, P_0\)\$](#) , where c_p^0 is a constant defined in McDougall (2003) and IOC et al. (2010).

2.3. Passage relations for “heat” variables

How entropy, potential temperature and Conservative Temperature are related to each other has been previously discussed in Tailleux (2010a) building upon previous work by Bacon and Fofonoff (1996) and McDougall (2003)

134 (see also IOC et al. (2010)). All the necessary relations are usually obtained
 135 from the total differential of the specific enthalpy h (also often referred to as
 136 the fundamental relation of thermodynamics), viz.

$$dh = Td\eta + \mu dS + v dP, \quad (9)$$

137 e.g., IOC et al. (2010), which can alternatively be written in terms of
 138 temperature, salinity and pressure as follows:

$$dh = c_p dT + \left(\mu - T \frac{\partial \mu}{\partial T} \right) dS + v(1 - \alpha T) dP, \quad (10)$$

139 where c_p is the specific heat capacity at constant pressure, μ is the relative
 140 chemical potential of seawater, $v = 1/\rho$ is the specific volume and α is the
 141 thermal expansion coefficient. The passage from (9) to (10) follows from the
 142 fact that the total differential of specific entropy in terms of temperature,
 143 salinity and pressure is given by:

$$d\eta = \frac{c_p}{T} dT - \frac{\partial \mu}{\partial T} dS - \frac{\alpha}{\rho} dP, \quad (11)$$

144 e.g., Tailleux (2010a). How the term $Td\eta + \mu dS$ in the enthalpy differential
 145 (9) transforms in the (θ, S) and (Θ, S) representations is given by Eqs. (B.2)
 146 and (B.3) of Tailleux (2010a) and in Appendix A.12 of IOC et al. (2010),
 147 specifically

$$\mu dS + T d\eta = \left(\mu - T \frac{\partial \mu_R}{\partial \theta} \right) dS + \frac{T c_p^R}{\theta} d\theta, \quad (12)$$

148

$$\mu dS + T d\eta = \left(\mu - \frac{T \mu_R}{\theta} \right) dS + \frac{T c_p^0}{\theta} d\Theta, \quad (13)$$

149 where $\mu_R = \mu(\theta, S, P_0)$ and $c_p^R = c_p(\theta, S, P_0)$. These relations were first de-
 150 rived by Bacon and Fofonoff (1996) and McDougall (2003) respectively.

Also useful are relations allowing to pass from the (θ, S) to (Θ, S) representation, which can be obtained by eliminating η between (12) and (13), which yields:

$$d\Theta = \frac{1}{c_p^0} \left(\mu_R - \theta \frac{\partial \mu_R}{\partial \theta} \right) dS + \frac{c_p^R}{c_p^0} d\theta, \quad (14)$$

$$d\theta = \frac{c_p^0}{c_p^R} d\Theta - \frac{1}{c_p^R} \left(\mu_R - \theta \frac{\partial \mu_R}{\partial \theta} \right) dS. \quad (15)$$

These two equations correspond to Eqs. (A.14) and (A.15) in Tailleux (2010a).

2.4. Implications for the evolution equations of the heat variables

The simplest way to obtain explicit expressions for the fluxes and non-conservative terms entering the evolution equations for entropy, potential temperature and Conservative Temperature (4)-(6) is to deduce the latter from the evolution equation for enthalpy, which energy considerations impose to be of the following form

$$\rho \frac{Dh}{Dt} = -\nabla \cdot [\rho \underbrace{(\mathbf{F}_h + \mathbf{F}_{rad} + \mathbf{F}_{oa})}_{\mathbf{F}_{htot}}] + \rho \varepsilon_K + \frac{DP}{Dt}. \quad (16)$$

In the above, $\mathbf{F}_h$ represents the enthalpy flux due to the molecular diffusive fluxes of heat and salt, $\mathbf{F}_{rad}$ represents represent the enthalpy flux due to incoming shortwave radiation and outgoing/downwelling long wave radiation, and $\mathbf{F}_{oa}$ is used to represent the decrease in ocean enthalpy following latent heat release associated with evaporation. Eq. (16) is similar to that given in IOC et al. (2010), except for the term $\mathbf{F}_{oa}$. By comparing (16) with the expression for the total derivative of enthalpy, viz.,

$$\frac{Dh}{Dt} = T \frac{D\eta}{Dt} + \mu \frac{DS}{Dt} + v \frac{DP}{Dt}, \quad (17)$$

171 it follows that

$$\rho \left(T \frac{D\eta}{Dt} + \mu \frac{DS}{Dt} \right) = -\nabla \cdot [\rho (\mathbf{F}_h + \mathbf{F}_{rad} + \mathbf{F}_{oa})] + \rho \varepsilon_K, \quad (18)$$

172 where ε_K is the viscous dissipation rate, which is related to the work against
 173 the stress tensor in the classical way, e.g., see Landau and Lifschitz (1987);
 174 Tailleux (2010b). Now, by combining (18) with the passage relations derived
 175 above and Eq. (3) for salinity, it is easy using elementary manipulations to
 176 obtain the results presented in the following paragraphs.

177 *Flux and non conservation of entropy.* First, (18) shows that the evolution
 178 equation for entropy can be written as

$$\rho \frac{D\eta}{Dt} = -\frac{\nabla \cdot (\rho \mathbf{F}_{htot})}{T} + \frac{\mu \nabla \cdot (\rho \mathbf{F}_S)}{T} + \frac{\rho \varepsilon_K}{T}. \quad (19)$$

179 This can be written in the generic form

$$\rho \frac{D\eta}{Dt} = -\nabla \cdot (\rho \mathbf{F}_\eta) + \rho \dot{\eta}_{irr}, \quad (20)$$

180 provided that $\mathbf{F}_\eta$ and $\dot{\eta}_{irr}$ are given by

$$\mathbf{F}_\eta = \frac{\mathbf{F}_{htot} - \mu \mathbf{F}_S}{T}, \quad (21)$$

181

$$\dot{\eta}_{irr} = -\mathbf{F}_S \cdot \nabla \left(\frac{\mu}{T} \right) + \mathbf{F}_{htot} \cdot \nabla \left(\frac{1}{T} \right) + \frac{\varepsilon_K}{T}. \quad (22)$$

182 *Flux and non conservation of Conservative Temperature.* By using (13) and
 183 (18), it follows that we have

$$\rho \left[\frac{T c_p^0}{\theta} \frac{D\Theta}{Dt} + \left(\mu - \frac{T \mu_R}{\theta} \right) \frac{DS}{Dt} \right] = -\nabla \cdot (\rho \mathbf{F}_{htot}) + \rho \varepsilon_K. \quad (23)$$

184 After some manipulation, it is possible to rewrite this equation in the generic
 185 form

$$\rho \frac{D\Theta}{Dt} = -\nabla \cdot (\rho \mathbf{F}_\Theta) + \rho \dot{\Theta}_{irr}, \quad (24)$$

186 provided that $\mathbf{F}_\Theta$ and $\dot{\Theta}_{irr}$ are given by

$$\mathbf{F}_\Theta = \frac{\theta}{c_p^0 T} \left[\mathbf{F}_{htot} - \left(\mu - \frac{T\mu_R}{\theta} \right) \mathbf{F}_S \right], \quad (25)$$

187

$$\dot{\Theta}_{irr} = \frac{\theta}{T c_p^0} \left[-\mathbf{F}_S \cdot \nabla \left(\mu - \frac{T\mu_R}{\theta} \right) - \mathbf{F}_\Theta \cdot \nabla \left(\frac{T c_p^0}{\theta} \right) + \varepsilon_K \right]. \quad (26)$$

188 *Flux and non conservation of potential temperature.* By using (12) and (18),
189 it follows that we have

$$\rho \left[\frac{T c_p^R}{\theta} \frac{D\theta}{Dt} + \left(\mu - T \frac{\partial \mu_R}{\partial \theta} \right) \frac{DS}{Dt} \right] = -\nabla \cdot (\rho \mathbf{F}_{htot}) + \rho \varepsilon_K. \quad (27)$$

190 After some manipulation, it is possible to rewrite this equation in the generic
191 form

$$\rho \frac{D\theta}{Dt} = -\nabla \cdot (\rho \mathbf{F}_\theta) + \rho \dot{\theta}_{irr}, \quad (28)$$

192 provided that $\mathbf{F}_\theta$ and $\dot{\theta}_{irr}$ are given by

$$\mathbf{F}_\theta = \frac{\theta}{c_p^R T} \left[\mathbf{F}_{htot} - \left(\mu - T \frac{\partial \mu_R}{\partial \theta} \right) \mathbf{F}_S \right], \quad (29)$$

193

$$\dot{\theta}_{irr} = \frac{\theta}{T c_p^R} \left[-\mathbf{F}_S \cdot \nabla \left(\mu - T \frac{\partial \mu_R}{\partial \theta} \right) - \mathbf{F}_\theta \cdot \nabla \left(\frac{T c_p^R}{\theta} \right) + \varepsilon_K \right]. \quad (30)$$

194 It is of interest to examine the implications of the above relations for the
195 form of the fluxes of θ and Θ at the surface. Evaluating (25) and (29) at
196 $z = 0$ yields respectively:

$$\mathbf{F}_\Theta = \frac{1}{c_p^0} (\mathbf{F}_h + \mathbf{F}_{rad} + \mathbf{F}_{oa}), \quad \text{at } z = 0, \quad (31)$$

197

$$\mathbf{F}_\theta = \frac{1}{c_p^R} (\mathbf{F}_q + \mathbf{F}_{rad} + \mathbf{F}_{oa}), \quad \text{at } z = 0, \quad (32)$$

198 where

$$\mathbf{F}_q = \mathbf{F}_h - \left(\mu - T \frac{\partial \mu}{\partial T} \right) \mathbf{F}_S \quad (33)$$

199 is a reduced heat flux that is discussed extensively below, by noting that
 200 $\mu = \mu_R$ at $z = 0$. The implications for the appropriate boundary conditions
 201 for θ and Θ are discussed in Section 5.

202 **3. Linking heat non-conservation to ocean surface properties**

203 An important difficulty with the form of the nonconservative terms for
 204 potential temperature, Conservative Temperature and entropy derived by
 205 Tailleux (2010a) and Graham and McDougall (2013) is that they rely on
 206 using explicit turbulent mixing parameterisations, which remain poorly con-
 207 strained. The purpose of this section is to show that $\dot{\eta}_{irr}$, $\dot{\theta}_{irr}$ and $\dot{\Theta}_{irr}$ can
 208 be related to measurable or derived properties provided that the oceans can
 209 be regarded as being in quasi steady state, which provides an independent
 210 mean to estimate such terms. To that end, it is important to first review
 211 the formulation of boundary conditions for heat, salt and freshwater for the
 212 ocean under the most general conditions.

213 *3.1. Boundary conditions for salt and freshwater*

214 The standard formulation of boundary conditions for salt and freshwater
 215 assumes that the latter move with different velocities $\mathbf{v}_s$ and $\mathbf{v}_w$ respectively,
 216 and that each satisfies a conservation equation of the form

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho \mathbf{v}_s) = 0, \quad (34)$$

$$\frac{\partial \rho_w}{\partial t} + \nabla \cdot (\rho \mathbf{v}_w) = 0. \quad (35)$$

218 where $\rho_s = \rho S$ and $\rho_w = \rho(1 - S)$ are the partial densities for salt and
 219 freshwater respectively, e.g., Warren (2006). According to non-equilibrium

thermodynamics, molecular diffusion of salt in solution arises from the difference between $\mathbf{v}_w$ and $\mathbf{v}_s$. The effect can be isolated by introducing the barycentric velocity $\mathbf{v} = S\mathbf{v}_s + (1 - S)\mathbf{v}_w$, which allows one to rewrite the above conservations equations as follows

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}) = -\nabla \cdot (\rho \mathbf{F}_S), \quad \frac{\partial \rho_w}{\partial t} + \nabla \cdot (\rho_w \mathbf{v}) = \nabla \cdot (\rho \mathbf{F}_S) \quad (36)$$

where the salt flux $\mathbf{F}_S$ is defined by

$$\rho \mathbf{F}_S = \rho_s(\mathbf{v}_s - \mathbf{v}) = \rho S(1 - S)(\mathbf{v}_s - \mathbf{v}_w), \quad (37)$$

which establishes that salt flux is only nonzero when $\mathbf{v}_s$ and $\mathbf{v}_w$ are different. The boundary conditions for $\mathbf{v}_s$ and $\mathbf{v}_w$ have been discussed by several authors, e.g., Warren (2006), IOC et al. (2010); if one assumes the sea surface height to be given by an equation of the form $z = \zeta(x, y, t)$ (which neglects spray and gravity waves overturns), they take the form

$$\frac{\partial \zeta}{\partial t} + \mathbf{u}_s \cdot \nabla_h \zeta - w_s = 0, \quad (38)$$

$$\rho_w \left[\frac{\partial \zeta}{\partial t} + \mathbf{u}_w \cdot \nabla_h \zeta - w_w \right] = \rho_f(P - E), \quad (39)$$

where $\rho_w = \rho(T, S, p)(1 - S)$ is the partial density of freshwater in seawater, while $\rho_f = \rho(T, 0, p)$ is the density of freshwater. Physically, the condition for salt assumes that no salt leaves the ocean, which is an idealisation, while that for freshwater assumes that the latter enters and leaves the ocean through precipitation P and evaporation E . Also useful is the boundary condition for the velocity difference $\mathbf{v}_s - \mathbf{v}_w$, obtained by taking the difference between (38) and (39),

$$(\mathbf{u}_s - \mathbf{u}_w) \cdot \nabla_h \zeta - (w_s - w_w) = \frac{\rho_f(P - E)}{\rho_w}, \quad (40)$$

238 which in turn implies the following boundary condition for the salt flux

$$\rho \mathbf{F}_S \cdot \mathbf{n} d\Sigma = \rho S(1 - S)(\mathbf{v}_s - \mathbf{v}_w) \cdot \mathbf{n} d\Sigma = -\rho_f S(E - P) dA, \quad (41)$$

239 where $dA = dx dy$ is the flat areal surface element, while $d\Sigma = \sqrt{1 + \|\nabla_h \zeta\|^2} dA$
 240 is the elemental area normal to the outward unit vector $\mathbf{n}$, e.g., see Beron-
 241 Vera et al. (1999).

242 3.2. Boundary conditions for heat

243 The formulation of the boundary conditions for the surface enthalpy flux
 244 due to radiation and latent heat release is straightforward, and given by

$$-\rho \mathbf{F}_{rad} \cdot \mathbf{n} d\Sigma = (Q_{sw} + Q_{lw}) dA \quad (42)$$

245

$$-\rho \mathbf{F}_{oa} \cdot \mathbf{n} d\Sigma = Q_{lh} dA \quad (43)$$

246 where Q_{sw} is the incoming shortwave radiation, Q_{lw} is the sum of the outgoing
 247 and incoming downwelling long wave radiation, and $Q_{lh} = -LE < 0$ is the
 248 latent heat flux, where $L = h_v - h_w$ is the latent heat flux, defined as the
 249 difference between the partial enthalpy of water vapour in moist air minus
 250 the partial enthalpy of freshwater in seawater, e.g., see Eq. (3.39.7) of IOC
 251 et al. (2010).

252 The proper formulation of the boundary condition for the diffusive flux
 253 of enthalpy $\mathbf{F}_h$ requires some care, as the latter is a priori affected by both
 254 salt and heat diffusion, but only the thermal part is related to the sensible
 255 heat flux. The way to remove the effects of salt diffusion can be achieved
 256 by introducing the reduced heat flux $\mathbf{F}_q$ that captures the effects of heat
 257 diffusion alone (assuming cross diffusive effects such as the Soret and Dufour

258 effects can be neglected), defined by

$$\mathbf{F}_q = \mathbf{F}_h - \left. \frac{\partial h}{\partial S} \right|_{T,P} \mathbf{F}_S = \mathbf{F}_h - \left(\mu - T \frac{\partial \mu}{\partial T} \right) \mathbf{F}_S \approx -\kappa c_p \nabla T, \quad (44)$$

259 where κ is the molecular diffusion of heat, e.g., Landau and Lifschitz (1987).

260 It is therefore $\mathbf{F}_q$, rather than $\mathbf{F}_h$, whose boundary condition is related to
261 the sensible heat flux, viz,

$$-\rho \mathbf{F}_q \cdot \mathbf{n} d\Sigma = Q_{sens} dA, \quad (45)$$

262 where Q_{sens} is the sensible heat flux.

263

264 3.3. Remarks on the conservative form of heat evolution equations

265 The study of volume-integrated budgets is facilitated by writing down
266 evolution equations in *conservative* form, which usually result from combining
267 the *advective* form of the equations with the mass conservation equation,
268 which is illustrated by the transformation

$$\rho \frac{Dq}{Dt} \rightarrow \frac{\partial(\rho q)}{\partial t} + \nabla \cdot (\rho q \mathbf{v}). \quad (46)$$

269 We argue, however, that the above conservative form (46) for the mass flux of
270 q is not optimal, since boundary conditions are more naturally formulated for
271 the salt and freshwater velocities $\mathbf{v}_s$ and $\mathbf{v}_w$ rather than for the barycentric
272 velocity $\mathbf{v}$; moreover, a process such as evaporation distillates the salt from
273 the freshwater part of seawater, which motivates us to write q in terms of its
274 partial salt and freshwater parts q_s and q_w as $q = Sq_s + (1 - S)q_w$, where q_s
275 and q_w are defined by

$$q_s = q + (1 - S) \frac{\partial q}{\partial S}, \quad q_w = q - S \frac{\partial q}{\partial S}, \quad (47)$$

276 e.g., IOC et al. (2010). This in turn implies

$$q_s - q_w = \frac{\partial q}{\partial S}, \quad (48)$$

where the partial derivative with respect to salinity is done at constant temperature and pressure. Making use of the above, as well as of the definition for the salt flux $\mathbf{F}_S = S(1 - S)(\mathbf{v}_s - \mathbf{v}_w)$, allows one to rewrite the advective enthalpy flux as follows

$$\begin{aligned} h\mathbf{v} &= [Sh_s + (1 - S)h_w]\mathbf{v} = \\ &= Sh_s\mathbf{v}_s + (1 - S)h_w\mathbf{v}_w + Sh_s(\mathbf{v} - \mathbf{v}_s) + (1 - S)h_w(\mathbf{v} - \mathbf{v}_w) \\ &= Sh_s\mathbf{v}_s + (1 - S)h_w\mathbf{v}_w - \frac{\partial h}{\partial S}\mathbf{F}_S. \end{aligned} \quad (49)$$

278 Using a similar approach allows one to rewrite the advective entropy flux as
279 follows

$$\eta\mathbf{v} = S\eta_s\mathbf{v}_s + (1 - S)h_w\mathbf{v}_w - \frac{\partial \eta}{\partial S}\mathbf{F}_S. \quad (50)$$

280 The main advantage of (49) and (50) is to elucidate the fact that the clas-
281 sical advective fluxes of enthalpy and entropy are actually made up of both
282 mass and diffusive fluxes, which is not a priori obvious and rarely discussed
283 (Warren (2006) alludes to it, but not very clearly). Next, we make use of
284 standard thermodynamic relations to link $\partial h/\partial S$ and $\partial \eta/\partial S$ to the relative
285 chemical potential μ and its temperature derivative as follows

$$\left. \frac{\partial h}{\partial S} \right|_{T,P} = \mu - T \frac{\partial \mu}{\partial T}, \quad \left. \frac{\partial \eta}{\partial S} \right|_{T,P} = - \frac{\partial \mu}{\partial T}, \quad (51)$$

286 which we then use to rewrite the sum of the advective and diffusive fluxes of
287 enthalpy and entropy as follows

$$h\mathbf{v} + \mathbf{F}_h = Sh_s\mathbf{v}_s + (1 - S)h_w\mathbf{v}_w + \mathbf{F}_q, \quad (52)$$

288

$$\eta \mathbf{v} + \frac{\mathbf{F}_h - \mu \mathbf{F}_S}{T} = S \eta_s \mathbf{v}_s + (1 - S) \eta_w \mathbf{v}_w + \frac{\mathbf{F}_q}{T}. \quad (53)$$

289 These relations are important and more useful, because they more clearly link
 290 the advective and diffusive parts of the fluxes to their boundary conditions by
 291 removing salt diffusion effects entirely. As a result, it is possible to rewrite
 292 the conservative form of the enthalpy and entropy evolution equations as
 293 follows

$$\frac{\partial(\rho\eta)}{\partial t} + \nabla \cdot [\rho_s \eta_s \mathbf{v}_s + \rho_w \eta_w \mathbf{v}_w] + \nabla \cdot \left[\rho \left(\frac{\mathbf{F}_q + \mathbf{F}_{rad} + \mathbf{F}_{oa}}{T} \right) \right] = \rho \dot{\eta}_{irr}, \quad (54)$$

294

$$\frac{\partial(\rho h)}{\partial t} + \nabla \cdot [\rho_s h_s \mathbf{v}_s + \rho_w h_w \mathbf{v}_w] + \nabla \cdot [\rho(\mathbf{F}_q + \mathbf{F}_{rad} + \mathbf{F}_{oa})] = \rho \varepsilon_K + \frac{DP}{Dt}, \quad (55)$$

295 which, as discussed below, greatly facilitates the understanding of the global
 296 budgets of the heat variables.

297 3.4. Insights from global budgets

298 Having clarified the nature of the boundary conditions for freshwater,
 299 salt, and heat, it is straightforward to show that the temporal evolution
 300 of the volume-integrated entropy, Conservative Temperature and potential
 301 temperature must be given by

$$\frac{d}{dt} \int_V \rho \eta dV = \int_S \frac{Q_{net}}{T} dA + \int_V \rho \dot{\eta}_{irr} + \int_S \rho_f \eta_w (P - E) dA, \quad (56)$$

302

$$\frac{d}{dt} \int_V \rho \Theta dV = \int_S \frac{Q_{net}}{c_p^0} + \int_V \rho \dot{\Theta}_{irr} dV + \int_S \frac{\rho_f h_w (P - E)}{c_p^0} dA, \quad (57)$$

303

$$\frac{d}{dt} \int_V \rho \theta dV = \int_S \frac{Q_{net}}{c_p^R} dA + \int_V \rho \dot{\theta}_{irr} dV + \int_S \rho_f T_s (P - E) dA, \quad (58)$$

304 where T_s is the ocean surface temperature, $Q_{net} = Q_{sens} + Q_{sh} + Q_{lw} + Q_{lh}$
 305 is the sum of all heat flux components. [Note also that the symbol \$S\$ in the](#)

306 integral refers to a surface integral and is not to be confused with salinity
 307 as in most of the paper. These expressions show that the volume integral of
 308 each quantity involves a term related to the net downward heat flux Q_{net} , a
 309 nonconservative term related to irreversible diffusive and viscous effects, and
 310 a mass flux term related to evaporation and precipitation (run-off is assumed
 311 to be included into precipitation, and will not be explicitly mentioned again).
 312 By considering a sufficiently long time average of the above budgets that
 313 approximately statistically steady-state conditions can be assumed to hold,
 314 the following constraints on the nonconservative terms are obtained

$$\int_V \rho \dot{\eta}_{irr} dV \approx - \int_S \frac{Q_{net}}{T} dA - \int_S \rho_f \eta_w (P - E) dA \quad (59)$$

315

$$\int_V \rho \dot{\Theta}_{irr} dV \approx - \int_S \frac{Q_{net}}{c_p^0} dA - \int_S \frac{\rho_f h_w (P - E)}{c_p^0} dA \quad (60)$$

316

$$\int_V \rho \dot{\theta}_{irr} dV \approx - \int_S \frac{Q_{net}}{c_p^R} dA + \int_S \rho_f T_s (P - E) dA. \quad (61)$$

317 Assuming that precipitation (including run-off) balances evaporation globally,
 318 each of the term involving $(P - E)$ can be written in the form

$$\int_S \rho_f Q (P - E) dA \approx \Delta Q M_E \quad (62)$$

319 where M_E is the total mass flux due to either precipitation or evaporation
 320 (assuming the two balance) in kg/s, and $\Delta Q = Q_P - Q_E$ is the difference
 321 between a representative value of Q for precipitation and Q_E a representa-
 322 tive value for evaporation. As mentioned in Griffies et al (2009), the term
 323 involving precipitation and evaporation is usually found to be subdominant
 324 in the heat budget, and is therefore neglected in the rest of the paper, but
 325 could be easily retained in a more rigorous analysis (although not necessarily

easy to estimate precisely). As a result, we approximate the above integrals as follows

$$\int_V \rho \dot{\eta}_{irr} dV \approx - \int_S \frac{Q_{net}}{T} dS, \quad (63)$$

$$\int_V \rho \dot{\Theta}_{irr} dV \approx - \int_S \frac{Q_{net}}{c_p^0} dS. \quad (64)$$

$$\int_V \rho \dot{\theta}_{irr} dV \approx - \int_S \frac{Q_{net}}{c_p^R} dS, \quad (65)$$

Before examining the implications of the above results, it is useful to recall the constraints on the net heat flux that needs to hold for a steady-state ocean following from the global conservation of energy, as well as from the balance equation for mechanical energy. To that end, it is first useful to write the net flux $Q_{total} = \int_S Q_{net} dA = Q_{in} - Q_{out}$ as the difference between a net positive input of heat minus a net cooling term. For a steady-state ocean, $Q_{out} \neq Q_{in}$ because the existence of mechanical sources of energy (due to the wind, tides, atmospheric pressure work, ...) implies that the total energy budget is given by:

$$Q_{in} - Q_{out} + W_{mech} = 0 \quad (66)$$

where W_{mech} denotes the power input by the mechanical sources of energy, see Tailleux (2010b) for more details on how to arrive at this result. Another useful constraint is obtained from the budget of mechanical energy (i.e., the sum of kinetic energy and gravitational potential energy), which leads to:

$$W_{mech} + \underbrace{\int_V P \frac{Dv}{Dt} dm}_B = \underbrace{\int_V \rho \varepsilon_K dV}_{D(KE)}, \quad (67)$$

where the term B represents the classical thermodynamic work of expansion/contraction, whereas $D(KE)$ denotes the total viscous dissipation.

345 3.5. *A priori estimate of nonconservative production of Θ*

346 Because the nature of its non conservation is quite different from that
 347 of entropy and potential temperature, we discuss Conservative Temperature
 348 first. By combining (64) and (66), it follows that:

$$\int_V \rho \dot{\Theta}_{irr} dV \approx -\frac{Q_{total}}{c_p^0} = \frac{Q_{out} - Q_{in}}{c_p^0} = \frac{W_{mech}}{c_p^0} > 0, \quad (68)$$

349 which states that the total nonconservative production of Conservative Tem-
 350 perature actually measures the overall power input due to the mechanical
 351 sources of energy, which is expected to be strictly positive. As established
 352 previously, the nonconservative production of Θ is the sum of two parts, one
 353 related to molecular diffusive processes, one related to viscous dissipation.
 354 We are primarily interested in estimating the former, which is the part pri-
 355 marily discussed in Tailleux (2010a) and Graham and McDougall (2013),
 356 and which we denote by $\dot{\Theta}_{irr}^{diff}$. This leads us to rewrite (68) as follows:

$$\int_V \rho \dot{\Theta}_{irr}^{diff} dV + \int_V \frac{\theta}{T c_p^0} \rho \varepsilon_K dV = \frac{W_{mech}}{c_p^0}. \quad (69)$$

357 Now, from the mechanical energy balance (67), we can eliminate W_{mech} in
 358 favour of the overall compressible work of expansion/contraction B and total
 359 viscous dissipation $D(KE)$, which yields:

$$\int_V \rho \dot{\Theta}_{irr}^{diff} dV = -\frac{B}{c_p^0} + \int_V \left(\frac{T - \theta}{T} \right) \frac{\rho \varepsilon_K}{c_p^0} dV \approx -\frac{B}{c_p^0}, \quad (70)$$

360 where the term involving viscous dissipation can be neglected owing to the
 361 fact that $(T - \theta)/T \ll 1$ is very small in the oceans (This approximation
 362 also assumes that B is of the same order of magnitude as $D(KE)$, which
 363 seems confirmed by published results about the mechanical energy budget of

numerical ocean models discussed further in the text). It also neglects the geothermal flux, which according to Graham and McDougall (2013) could potentially significantly alter the result and the estimation of the nonconservation of CT. Here, it is neglected on the grounds that OGCMs still do not systematically include it.. Eq. (70) is an important result, which states that the diffusive part of the nonconservative production of Conservative Temperature is a direct measure of the overall work of expansion/contraction, a result previously obtained by Tailleux (2012).

Although the precise magnitude of B and of compressible effects in the oceans is still a matter of debate, e.g., Tailleux (2009a), let us note that in the context of ocean Boussinesq modelling, this term is classically approximated by substituting the pressure by the Boussinesq pressure $P \rightarrow -\rho_0 g z$, the mass element by the Boussinesq mass element $dm \rightarrow \rho_0 dV$, using the Taylor series expansion $v \approx (1/\rho_0) - (\rho - \rho_0)/\rho_0^2$ it follows that $PDv/Dt \approx (-\rho_0 g z)(-D\rho/Dt)/\rho_0^2$, which yields:

$$\int_V P \frac{Dv}{Dt} dm \approx \int_V \rho_0 g z \frac{1}{\rho_0^2} \frac{D\rho}{Dt} \rho_0 dV = \int_V g z \frac{D\rho}{Dt} dV. \quad (71)$$

In Boussinesq ocean models with a realistic nonlinear equation of state, density is nonconservative, and obeys an equation of the form:

$$\frac{D\rho}{Dt} = \nabla \cdot \mathbf{F}_\rho + \dot{\rho}_{irr}, \quad (72)$$

where $\mathbf{F}_\rho$ is the diffusive flux of density due to the turbulent mixing of temperature and salinity, whereas $\dot{\rho}_{irr}$ represents the effects due to the nonlinearities of the equation of state (Including compressibility effects, assumed to be small relative to the effects of cabelling and thermobaricity). This term is dominated by cabelling in the upper stratified ocean, but by thermobaricity

386 in the weekly stratified abyssal ocean, as discussed by Oliver and Tailleux
 387 (2013). Inserting (72) into (71) allows the latter to be rewritten as:

$$\int_V gz \frac{D\rho}{Dt} dV \approx \underbrace{\int_V \rho_0 K_v N^2 dV}_{>0} + \underbrace{\int_V gz \dot{\rho}_{irr} dV}_{<0}. \quad (73)$$

388 The first term is positive and is associated with the classical result that mix-
 389 ing raises the centre of gravity of a fluid with a linear equation of state. The
 390 second term is in general dominated by cabelling and associated with contrac-
 391 tion upon mixing, which plays a dominant role in the ocean energy budget,
 392 as perhaps first discussed by Gnanadesikan et al. (2005). In a steady-state,
 393 (71) can equivalently be rewritten in the following three equivalent forms:

$$\int_V gz \frac{D\rho}{Dt} dV = - \int_V \mathbf{u} \cdot \nabla_h P dV = - \int_V \rho g w dV \quad (74)$$

394 which can all provide the basis for estimating B using OGCM results, as
 395 reviewed in Tailleux (2013). The first estimation of B for a realistic ocean
 396 model configuration is perhaps due to Toggweiler and Samuels (1998), based
 397 on the last expression in (74), who were the first to suggest that B is negative
 398 in the ocean, rather than positive, in contrast to what was previously hypoth-
 399 esised by Oort et al (1994). Specifically, they find for the volume-averaged
 400 value of $B \approx 1.12 \times 10^{-6} \text{erg cm}^{-3} \text{s}^{-1} = 1.12 \times 10^{-13} \text{J} \times 10^6 \text{m}^{-3} \text{s}^{-1} = 1.12 \times$
 401 10^{-7}Wm^{-3} . The ocean volume is about $V_{oc} = 1.3 \times 10^6 \text{km}^3 = 1.3 \times 10^{18} \text{m}^3$,
 402 resulting in a net energy conversion of $1.456 \times 10^{11} \text{W} = 0.14 \text{TW}$. In Gre-
 403 gory and Tailleux (2011), the value of B in HadCM3 control climate is
 404 -0.494TW , based on the second expression in (74), while in the low resolu-
 405 tion version FAMOUS it is -0.060TW . In $4 \times \text{CO}_2$ control climate, estimates
 406 of B in both FAMOUS and HadCM3 are $O(0.12 - 0.13 \text{TW})$ (and negative),
 407 and hence similar to Toggweiler and Samuels (1998) estimates.

408 3.6. *A priori estimates of nonconservative entropy production*

The second law of thermodynamics imposes the nonconservative production of entropy by molecular diffusive processes and viscous processes to be strictly positive. In a steady-state ocean, internal entropy production must be furthermore balanced by [export](#) of entropy by surface heat and freshwater fluxes. This can be expressed as:

$$\begin{aligned}
 \int_V \rho \dot{\eta}_{irr} dV &\approx - \int_S \frac{Q_{net}}{T} dS = \frac{Q_{out}}{T_{out}} - \frac{Q_{in}}{T_{in}} \\
 &= \frac{Q_{in} + W_{mech}}{T_{out}} - \frac{Q_{in}}{T_{in}} = \frac{1}{T_{out}} \left[\left(\frac{T_{in} - T_{out}}{T_{in}} \right) Q_{in} + W_{mech} \right], \quad (75)
 \end{aligned}$$

409 where $1/T_{in}$ is the weighted average of $1/T$ restricted to regions where $Q_{net} >$
 410 0 , and $1/T_{out}$ the weighted average of $1/T$ over regions where $Q_{net} < 0$. In the
 411 ocean, the mechanical power input W_{mech} is strictly positive, while heating
 412 takes place on average at higher temperatures than cooling, as is expected
 413 from a heat engine (see related discussion by Tailleux (2010b)) so that
 414 $(T_{in} - T_{out})Q_{in} > 0$. As a result, the right-hand side of (75) is also strictly
 415 positive, and therefore in agreement with the second law.

417 As for Conservative Temperature, it is useful to separate the diffusive and
 418 viscous contributions to the nonconservative entropy production, viz.,

$$\int_V \rho \dot{\eta}_{irr} dV = \int_V \rho \dot{\eta}_{irr}^{diff} dV + \int_V \frac{\rho \varepsilon_K}{T} dV = \int_V \rho \dot{\eta}_{irr}^{diff} dV + \frac{D(KE)}{T_\varepsilon}. \quad (76)$$

where the ‘viscous’ temperature T_ε is defined so as to make the above equality exact, e.g., see Tailleux (2010b). By using the mechanical energy balance $W_{mech} + B = D(KE)$, we can eliminate W_{mech} from Eq. (75) in favour of B and $D(KE)$ to obtain the following expression for the diffusive part of the

irreversible entropy production:

$$\begin{aligned} \int_V \rho \dot{\eta}_{\text{irr}}^{\text{diff}} dV &= \left(\frac{T_{\text{in}} - T_{\text{out}}}{T_{\text{in}} T_{\text{out}}} \right) Q_{\text{in}} - \frac{B}{T_{\text{out}}} + \left(\frac{1}{T_{\text{out}}} - \frac{1}{T_{\varepsilon}} \right) D(KE) \\ &\approx \frac{1}{T_{\text{out}}} \left[\left(\frac{T_{\text{in}} - T_{\text{out}}}{T_{\text{in}}} \right) Q_{\text{in}} + c_p^0 \int_V \rho \dot{\Theta}_{\text{irr}}^{\text{diff}} dV \right]. \end{aligned} \quad (77)$$

To arrive at (77), we neglected the term proportional to the overall viscous dissipation $D(KE)$ relative to the term proportional to Q_{in} , while we replaced the compressible thermodynamic work term B by its expression in terms of the nonconservative production of potential enthalpy. Eq. (77) is a useful result, which helps understand the link between the nonconservative production of potential enthalpy and entropy. One important remark of McDougall (2003) and Graham and McDougall (2013) is that entropy is considerably more nonconservative than potential enthalpy; in order for this to be true, Eq. (77) requires that the following inequality be satisfied:

$$T_{\text{out}} \int_V \rho \dot{\eta}_{\text{irr}}^{\text{diff}} \approx \left(\frac{T_{\text{in}} - T_{\text{out}}}{T_{\text{in}}} \right) Q_{\text{in}} \gg \left| c_p^0 \int_V \rho \dot{\Theta}_{\text{irr}}^{\text{diff}} dV \right|. \quad (78)$$

We can use published estimates of entropy production by surface heat fluxes to convince oneself that this inequality must indeed be satisfied in the ocean. For instance, Pascale et al (2011) estimate that the average entropy production is $O(1 \text{ mW.K}^{-1}.\text{m}^{-2})$ and hence that the total entropy production is of the order of $3.10^{11} \text{ W.K}^{-1}$, using the result that the total area of the ocean is approximately 3.10^{14} m^2 . Using $T_{\text{out}} \approx 285 \text{ K}$, the consequence is that $(T_{\text{in}} - T_{\text{out}})Q_{\text{in}}/T_{\text{out}}$ must be of the order $85.5 \times 10^{12} \text{ W} = 85.5 \text{ TW}$. This value is in between one and two orders of magnitude larger than estimates for W_{mech} , which confirms that it is therefore expected to be much larger than estimates for B and hence of potential enthalpy nonconservative production.

439 This is consistent with McDougall (2003) conclusion that nonconservative
 440 production of potential enthalpy is about 2 orders of magnitude smaller than
 441 the nonconservative production of entropy scaled by T_{out} . The above shows
 442 that such a conclusion can be arrived at using much simpler arguments based
 443 on global budgets. A caveat should be mentioned, however, which is due to
 444 the fact that in contrast to entropy or Conservative Temperature, whose
 445 nonconservation is usually sign-definite, the nonconservation of θ can be of
 446 either sign. Since our approach focuses on global budgets, it will therefore
 447 underestimates the nonconservation of of potential temperature as compared
 448 to Graham and McDougall (2013), which focuses on the root-mean square
 449 of locally estimated nonconservation terms.

450 Note that our approach here is very different from Yan et al. (2004),
 451 who estimate the irreversible entropy production due to radiative heat fluxes
 452 to be at least two orders of magnitude greater than the one discussed in
 453 Pascale et al (2011). The difference arises because Yan et al. (2004)
 454 considers that upon thermalisation with the ocean, the entropy of radiation
 455 increases from its very low value F_{sw}/T_{sun} upon leaving the sun to the very
 456 high value F_{sw}/T_{ocean} upon thermalisation with matter (i.e, seawater here),
 457 where T_{sun} and T_{ocean} are the temperatures of the sun and ocean surface
 458 respectively. As a result, their entropy budget is dominated by terms like
 459 $F_{sw}(1/T_{sun} - 1/T_{ocean})$, but arguably, this irreversible production term is
 460 better viewed as part of the sun+photons+earth system, rather than as part
 461 of the ocean. See also Pelkowski (2014) for a recent discussion of the entropy
 462 of radiation. The viewpoint taken here is that upon thermalisation, photons
 463 lose their identity, and just contribute to increase the energy levels of the

464 matter with which they interact. This interaction is most easily treated as
 465 regular 'heating', and does not require the knowledge of the initial entropy
 466 of the photons at the time they were emitted by the sun.

467 3.7. *A priori estimates of nonconservative production of θ*

We now apply the above ideas to the problem of deriving a priori estimates for the nonconservative production of potential temperature, which leads us to rewrite (65) as follows:

$$\begin{aligned}
 & \int_V \rho \dot{\theta}_{irr} dV = - \int_S \frac{Q_{net}}{c_p^R} dS = \frac{Q_{out}}{c_{p,out}^R} - \frac{Q_{in}}{c_{p,in}^R} \\
 & = \frac{Q_{in} + W_{mech}}{c_{p,out}^R} - \frac{Q_{in}}{c_{p,in}^R} = \frac{1}{c_{p,out}^R} \left[\left(\frac{c_{p,in}^R - c_{p,out}^R}{c_{p,in}^R} \right) Q_{in} + W_{mech} \right], \quad (79)
 \end{aligned}$$

468 where $c_{p,in}^R$ and $c_{p,out}^R$ are the reciprocal of weighted means of $1/c_p^R$ averaged
 469 over the regions of net heating and cooling respectively, and defined so as
 470 to make the above decomposition exact. Eq. (79) shows that the nature of
 471 the potential temperature nonconservation is in many ways similar to that
 472 for entropy, given that (79) is essentially identical in structure to (75) with
 473 c_p^R in the former replacing T in the latter. As previously, we separate the
 474 nonconservative production of θ into a diffusive and viscous part, so that
 475 (79) becomes:

$$\int_V \rho \dot{\theta}_{irr}^{diff} dV + \frac{D(KE)}{c_{p,\varepsilon}^R} = \left(\frac{c_{p,in}^R - c_{p,out}^R}{c_{p,in}^R c_{p,out}^R} \right) Q_{in} + \frac{W_{mech}}{c_{p,out}^R}, \quad (80)$$

where we defined the quantity $c_{p,\varepsilon}^R$ via the relation:

$$\frac{\int_V \rho \varepsilon_K}{c_{p,\varepsilon}^R} = \int_V \frac{\theta}{T c_p^R} \rho \varepsilon_K dV.$$

As a result, we obtain the following expression for the diffusive part of $\dot{\theta}_{irr}$:

$$\begin{aligned} \int_V \rho \dot{\theta}_{irr}^{diff} dV &= \left(\frac{c_{p,out}^R - c_{p,in}^R}{c_{p,in}^R c_{p,out}^R} \right) Q_{in} - \frac{B}{c_{p,out}^R} + \left(\frac{1}{c_{p,out}^R} - \frac{1}{c_{p,\varepsilon}^R} \right) \int_V \rho \varepsilon_K \\ &\approx \frac{1}{c_{p,out}^R} \left[c_p^0 \int_V \rho \dot{\Theta}_{irr}^{diff} + \left(\frac{c_{p,in}^R - c_{p,out}^R}{c_{p,in}^R} \right) Q_{in} \right], \end{aligned} \quad (81)$$

where the approximation was obtained by neglecting the term proportional to viscous dissipation relative to the term proportional to Q_{in} . Eq. (81) is a key result of this paper, for it provides an explicit expression for the difference between the nonconservative production of potential and Conservative Temperature, which appears to be controlled by the net heating Q_{in} , as well as by the spatial variations of the heat capacity at the surface. One of McDougall (2003) key conclusions is that the overall nonconservative production of Conservative Temperature should be about two orders of magnitude smaller than the non-conservation of potential temperature. According to (81), this can be the case only if the following constraint is satisfied:

$$\left| c_{p,out}^R \int_V \rho \dot{\theta}_{irr}^{diff} dV \right| \approx \left| \left(\frac{c_{p,in}^R - c_{p,out}^R}{c_{p,in}^R} \right) Q_{in} \right| \gg \left| c_p^0 \int_V \rho \dot{\Theta}_{irr}^{diff} \right|. \quad (82)$$

Physically, this constraint requires that the weighted averaged heat capacity over heating regions be significantly different than the weighted averaged heat capacity over the cooling regions. Why this should be the case in the ocean is unclear, and the nature of the spatial variations in c_p^R giving rise to such a big difference is addressed empirically in the next section using observations. Eq. (81) shows that the exact difference between the net non-conservative production of potential and Conservative Temperature depends on the particular circumstances of the system studied, and is not entirely

496 an intrinsic property of the conservative versus potential temperature, in the
 497 sense that for a fluid with nearly constant heat capacity, potential tempera-
 498 ture would be nearly as conservative as Conservative Temperature. In fact,
 499 the net nonconservative production of θ could be even less than for Θ if the
 500 the surface ocean properties were such as to make the two terms within the
 501 square brackets in (81) cancel out. Whether such a configuration could oc-
 502 cur as a result of changes in the state of our climate is left as an intriguing
 503 open question, whose answer could perhaps give us important clues about
 504 the functioning of the ocean/atmosphere coupling.

505 **4. Observational constraints on nonconservative effects**

506 *4.1. Data sources*

507 In this section, we use freely available climatological datasets to estimate
 508 the two surface integrals

$$\begin{aligned}
 & - \int_S \frac{Q_{net}}{T} dA, \tag{83}
 \end{aligned}$$

$$\begin{aligned}
 & - \int_S \frac{Q_{net}}{c_p^R} dA, \tag{84}
 \end{aligned}$$

510 which we showed above to dominate the diffusive part of the nonconservative
 511 production of entropy and potential temperature. The estimation of these
 512 two integrals requires the knowledge of surface values of temperature and
 513 salinity (to estimate c_p^R), as well as estimates of the net heat flux into the
 514 oceans. The computations discussed in the following sections were obtained
 515 by using the annual mean surface temperature and salinity from the World
 516 Ocean Database 2013, as well as the balanced NOCS net heat flux product
 517 discussed in Grist and Josey (2003).

518 4.2. Observational constraints on irreversible entropy production

519 As mentioned by Grist and Josey (2003), most available heat flux prod-
 520 ucts are unbalanced. Even the balanced NOCS heat product version 1.1a
 521 depicted in Fig. 1, which is obtained through an inverse method is still un-
 522 balanced by -5 W.m^{-2} , which amounts a net cooling of about $-1.5 \cdot 10^{15} \text{ W} =$
 523 -1.5 PW . Were the oceans to be in a true steady-state and to satisfy the
 524 global energy balance $Q_{cooling} = Q_{in} + W_{mech}$, the net cooling should exceed
 525 the net heating so as to cancel out the Joule heating arising from the viscous
 526 dissipation of the power input due to the mechanical sources of energy. The
 527 net heating is so much larger than the power input by mechanical sources
 528 of energy ($Q_{in} = O(2 \text{ PW})$ versus $W_{mech} = O(1 - 10 \text{ TW})$) that it is very
 529 difficult to ascertain that the observed imbalance in available heat flux prod-
 530 ucts occur because of the need to balance W_{mech} , or as the result of the
 531 large uncertainties plaguing the evaluation of the various terms entering the
 532 heat budget. The NOCS heat flux product comes in two different versions,
 533 one that is unbalanced, and another one that was balanced using an inverse
 534 method.

535 To assess the role of imbalance in heat flux products, we write the net
 536 heat flux $Q_{net} = \bar{Q} + \tilde{Q}$ as the sum of a constant spatially uniform component
 537 $\bar{Q}$ plus a component $\tilde{Q}$ that integrates to zero. The entropy integral (83) thus
 538 becomes:

$$-\int_S \frac{Q_{net}}{T} dS = -\bar{Q} \int_S \frac{dS}{T} - \int_S \frac{\tilde{Q}}{T} dS = -\frac{A_{oc}\bar{Q}}{\bar{T}} - \frac{\tilde{Q}_{in}}{T_{in}} + \frac{\tilde{Q}_{out}}{T_{out}} \quad (85)$$

where $\bar{T}$ is the geometric average of the surface temperature, while $\tilde{Q}_{in}$ and
 $\tilde{Q}_{out}$ are now defined to balance exactly, i.e., to satisfy $\tilde{Q}_{in} = \tilde{Q}_{out}$, with A_{oc}

the surface area of the oceans. Here, the imbalance in the NOCS 1.1a heat flux product is of the order $\bar{Q} \approx 5 \text{ W.m}^{-2}$, so that

$$-\frac{A_{oc}\bar{Q}}{\bar{T}} \approx \frac{3.10^{14} \text{ m}^2 \times 5 \text{ W.m}^{-2}}{291} \approx 5.10^{12} \text{ W.K}^{-1}.$$

539 In the present case, the imbalance in the heat flux product is so large that
 540 if retained, it would be the term dominating the entropy budget. In reality,
 541 we expect the contribution from $\tilde{Q}_{in}$ and $\tilde{Q}_{out}$ to dominate.

542 We can try to anticipate the results of observational computation by using
 543 scaling argument, based on using the following typical values $\tilde{Q}_{in} = 2.10^{15} \text{ W}$,
 544 $T_{in} - T_{out} = 15^\circ\text{C}$, $T_{in}T_{out} \approx 300^2 \text{ K}^2$, which gives us

$$\frac{T_{in} - T_{out}}{T_{in}T_{out}}\tilde{Q}_{in} \approx \frac{15 \times 2.10^{15}}{300^2} = 0.33 \times 10^{12} \text{ W.K}^{-1}. \quad (86)$$

545 This value is equal to $1.1 \text{ mW.m}^{-2}.\text{K}^{-1}$ when divided by the area of the ocean,
 546 which is comparable to that estimated in the UK Hadley Centre coupled
 547 climate model HadCM3 by Pascale et al (2011). Physically, it is also useful
 548 to decompose the total temperature $T = \bar{T} + T'$ into a mean and perturbation
 549 part, in order to approximate entropy production as

$$-\int_S \frac{\tilde{Q}}{\bar{T}} dS \approx \frac{1}{\bar{T}^2} \int_S \tilde{Q}T' dS, \quad (87)$$

550 which shows that the diffusive part of entropy production is primarily con-
 551 trolled by the degree of correlation between the surface heat flux and the
 552 surface temperature anomalies. Fig. 2 shows the spatial map of the leading
 553 order term $-\tilde{Q}/\bar{T}$ and the second order term $\tilde{Q}T'/\bar{T}^2$ in entropy production.
 554 The leading order term is a rescaled version of the net heat flux depicted in
 555 Fig. 1. Numerical estimates yield $0.3 \times 10^{12} \text{ W.K}^{-1}$ for the total entropy pro-
 556 duction, which is nearly identical to the scaling argument above. Multiplying

557 this term by $\bar{T} = 291$ K yields 87 TW, which is the number to be compared
 558 with the non conservation of potential enthalpy, following McDougall (2003)
 559 and Graham and McDougall (2013).

560 4.3. *Observational constraints on irreversible production of θ*

561 Leaving out the unbalanced part of the heat flux, (84) becomes:

$$\int_V \rho \dot{\theta}_{irr}^{diff} dV \approx - \int_S \frac{\tilde{Q}}{c_p^R} dS = \frac{\tilde{Q}_{out}}{c_{p,out}^R} - \frac{\tilde{Q}_{in}}{c_{p,in}^R} = \left(\frac{1}{c_{p,out}^R} - \frac{1}{c_{p,in}^R} \right) \tilde{Q}_{in}, \quad (88)$$

562 so that fundamentally, the non conservation of potential temperature arises
 563 principally from the spatial variations of c_p^R due to the spatial variations of
 564 surface temperature and salinity. In this paper, we use the climatological
 565 annual mean values of temperature and salinity depicted in Fig. 3. The
 566 implied variations in c_p are depicted in the bottom panel of Fig. 4, while
 567 the top panel illustrates the dependence of c_p on temperature and salinity.
 568 The spatial distribution of the leading order contribution $-\tilde{Q}/c_p^R(\bar{S}, \bar{\theta}, P_0)$
 569 (where $c_p^R(\bar{S}, \bar{\theta}, P_0)$ is spatially uniform) is illustrated in the top left panel of
 570 Fig. 5, while the the second order contribution (due to the departure of c_p^R
 571 from its mean), is depicted in the top right panel. As in the case of entropy
 572 production, the leading order term is merely a rescaled version of the net
 573 heat flux illustrated in Fig. 1. Only the second order contribution, however,
 574 contributes to the surface integral, found to be $\bar{c}_p \int_V \rho \dot{\theta}_{irr}^{diff} \approx 4.6$ TW. In order
 575 to understand the physical origin of this number, it is useful to examine the
 576 relative contribution of the temperature and salinity dependence of c_p^R on the
 577 result. To that end, we can use a Taylor series expansion to write:

$$\frac{1}{c_p^R} \approx \frac{1}{\bar{c}_p} - \frac{1}{\bar{c}_p^2} \frac{\partial c_p}{\partial T}(\bar{T}, \bar{S}, P_0)(T - \bar{T}) - \frac{1}{\bar{c}_p^2} \frac{\partial c_p}{\partial S}(\bar{T}, \bar{S}, P_0)(S - \bar{S}) \quad (89)$$

578 where $\bar{c}_p = c_p(\bar{T}, \bar{S}, P_0)$, which in turns yields:

$$-\int_S \frac{\tilde{Q}}{c_p^R} dS \approx \frac{1}{\bar{c}_p^2} \frac{\partial \bar{c}_p}{\partial T} \int_S \tilde{Q} T' dS + \frac{1}{\bar{c}_p^2} \frac{\partial \bar{c}_p}{\partial S} \int_S \tilde{Q} S' dS. \quad (90)$$

579 Fig. 6 shows that c_p is more sensitive to salinity than to temperature. How-
 580 ever, the relative contribution of temperature and salinity anomalies in (90)
 581 is not a priori obvious, because even though the dependence of c_p on tem-
 582 perature is much smaller than on salinity, salinity anomalies do not correlate
 583 strongly with heat flux anomalies, in contrast to temperature anomalies.
 584 The numerical evaluation of the two terms in (90) supports this, with the
 585 temperature and salinity terms being approximately 3.6 TW and 1 TW re-
 586 spectively, thus showing that it is actually the c_p dependence on temperature
 587 that ultimately dominates, which differs from McDougall (2003).

588

589 5. Summary and implications for ocean modelling

590 The results of this paper make it clear that the non conservation of po-
 591 tential temperature and that of Conservative Temperature are fundamen-
 592 tally different, since the non conservation of the former primarily reflects
 593 its production/destruction by surface fluxes, whereas the non conservation
 594 of the latter is primarily a measure of the thermodynamic work of expan-
 595 sion/contraction (in a steady-state). As a result, the only way to preserve
 596 the balance between surface and interior production/destruction established
 597 for the non-averaged Navier-Stokes equations would require to replace the
 598 evolution equation and boundary conditions for potential temperature used

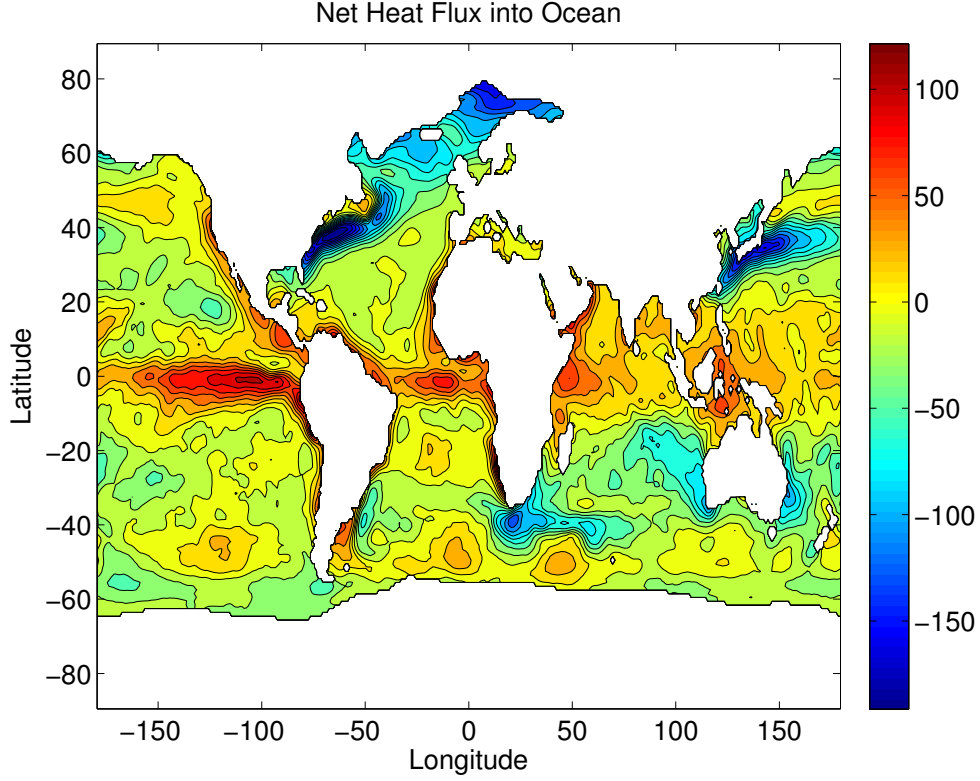


Figure 1: Net heat flux into the ocean in W.m^{-2} from the balanced NOCS 1.1a heat product.

599 in current OGCM formulations by the following ones

$$\frac{D\theta}{Dt} = -\nabla \cdot \mathbf{F}_\theta \quad \rightarrow \quad \frac{D\theta}{Dt} = -\frac{1}{c_p^R} \nabla \cdot (c_p^R \mathbf{F}_\theta) + N.C., \quad (91)$$

600

$$K \frac{\partial \theta}{\partial z}(z=0) = \frac{Q_{sens}}{\rho_0 c_{p0}} \quad \rightarrow \quad K \frac{\partial \theta}{\partial z}(z=0) = \frac{Q_{sens}}{\rho_0 c_p^R}, \quad (92)$$

601 where N.C. denotes additional nonconservative effects discussed below, with
 602 corresponding changes required for the radiative and latent heat fluxes, and
 603 K a vertical turbulent eddy diffusivity.

604 In order to fully specify the form of the potential temperature equation

605 (91), one needs a way to express its nonconservative part in terms of the tur-
 606 bulent fluxes of heat and salt. Tailleux (2010a) and Graham and McDougall
 607 (2013) both show that the form of the nonconservative part follows from
 608 assuming a certain quantity (in addition to salinity) to be conservative, but
 609 they disagree on which one. Specifically, Tailleux (2010a) assumes it to be
 610 total energy, as for the non-averaged Navier-Stokes equations, whereas Gra-
 611 ham and McDougall (2013) use a pseudo-conservative quantity — namely
 612 a locally-referenced form of potential enthalpy — which they take to vary
 613 from one grid-point to the next, so that although it is treated as conservative
 614 for the purpose of estimating the non conservation of θ and CT, it is actu-
 615 ally nonconservative from a strict mathematical viewpoint (hence referred to
 616 here as pseudo-conservative). Despite being based on different approaches,
 617 the expressions obtained by Graham and McDougall (2013) and Tailleux
 618 (2010a) are quite similar, the main difference being the former lacking the
 619 terms proportional to the pressure gradient of the latter.

620 For lack of definite understanding about how to handle the non conserva-
 621 tion of potential temperature, IOC et al. (2010) recommended that OGCMs
 622 should adopt Conservative Temperature and Absolute Salinity as their new
 623 prognostic variables, on the grounds that the non conservation of such quan-
 624 tities is sufficiently small to justify treating them as exactly conservative, and
 625 hence governed by

$$\frac{D\Theta}{Dt} = -\nabla \cdot \mathbf{F}_\Theta, \quad \frac{DS_A}{Dt} = -\nabla \cdot \mathbf{F}_S, \quad (93)$$

626 (with additional terms required to handle radiation and latent heat release
 627 in the CT equation). Turbulent fluxes are expressed in terms of a diffusivity

628 tensor $\mathbf{K}$, including diapycnal, mixing and eddy-induced transport, as follows

$$\mathbf{F}_S = -\mathbf{K}\nabla S_A, \quad \mathbf{F}_\Theta = -\mathbf{K}\nabla\Theta. \quad (94)$$

629 It is important to recognise, however, that switching to (Θ, S_A) as prognos-
 630 tic variables necessitates the specification of explicit functional relationships
 631 $T = T(\Theta, S_A, P)$ and $\theta = \theta(\Theta, S_A, P)$ allowing one to invert T and θ from the
 632 knowledge of Θ , S_A and P . Such inverse relations are available in the form of
 633 Matlab subroutines as part of the Gibbs Sea Water (GSW) Library (available
 634 at www.teos-10.org) called **gsw_t_from_CT.m** and **gsw_pt_from_CT.m**
 635 for instance. Since such inverse relations are known from basic thermody-
 636 namic principles, they can be differentiated in order to obtain $d\theta$ as a function
 637 of $d\Theta$ and dS_A , which are none other than the passage relations (14) and (15)
 638 derived previously. Thus, Eq. (15) yields the following equation for $D\theta/Dt$,

$$\frac{D\theta}{Dt} = \frac{c_p^0}{c_p^R} \frac{D\Theta}{Dt} - \frac{1}{c_p^R} \left(\mu_R - \theta \frac{\partial \mu_R}{\partial \theta} \right) \frac{DS_A}{Dt}, \quad (95)$$

639 while (14) yields the following expression for $\nabla\Theta$

$$\nabla\Theta = \frac{1}{c_p^0} \left(\mu_R - \theta \frac{\partial \mu_R}{\partial \theta} \right) \nabla S_A + \frac{c_p^R}{c_p^0} \nabla\theta. \quad (96)$$

640 It is now straightforward to combine (93), (94), (95) and (96) to deduce what
 641 the evolution equation for θ and turbulent flux $\mathbf{F}_\theta$ should be to be equivalent
 642 to the proposed IOC et al. (2010) (Θ, S_A) formulation, viz.,

$$\frac{D\theta}{Dt} = -\frac{1}{c_p^R} \nabla \cdot (c_p^R \mathbf{F}_\theta) - \frac{1}{c_p^R} \mathbf{F}_S \cdot \nabla \left(\mu_R - \theta \frac{\partial \mu_R}{\partial \theta} \right), \quad (97)$$

643

$$\mathbf{F}_\theta = -\mathbf{K}\nabla\theta. \quad (98)$$

As to the surface boundary condition for Θ , (14) shows that it should be

$$K \frac{\partial \Theta}{\partial z}(z=0) = \frac{Q_{sens}}{\rho_0 c_p^0} + \frac{1}{c_p^0} \left(\mu_R - \theta \frac{\partial \mu_R}{\partial \theta} \right) K \frac{\partial S_A}{\partial z}(z=0),$$

644

$$= \frac{Q_{sens}}{\rho_0 c_p^0} + \frac{1}{c_p^0} \left(\mu_R - \theta \frac{\partial \mu_R}{\partial \theta} \right) \frac{\rho_f S(E - P)}{\rho} \quad (99)$$

645 and hence that it should contain a term proportional to the salinity boundary
 646 condition, which is consistent with the fact that it is the reduced heat flux
 647 $\mathbf{F}_q = F_h - (\partial h / \partial S) \mathbf{F}_S$, rather than $\mathbf{F}_h$, which is linked to the sensible heat
 648 flux. This is an important point that is mentioned neither in McDougall
 649 (2003) nor in IOC et al. (2010), but which would require an additional
 650 modification to existing codes when switching to a (Θ, S_A) formulation. In
 651 order to assess the relative merits of the (θ, S) versus (Θ, S_A) formulations,
 652 we find it useful to separate the conservative and nonconservative part of the
 653 right-hand side of (97) as follows

$$\frac{D\theta}{Dt} = -\nabla \cdot \mathbf{F}_\theta + \dot{\theta}_{irr}. \quad (100)$$

The expression for the nonconservative term $\dot{\theta}_{irr}$ can be further clarified by
 expanding the gradients of the different functions of θ and S_A , leading to

$$\begin{aligned} \dot{\theta}_{irr} &= A(\theta, S_A) \nabla \theta^T \cdot (\mathbf{K} \nabla \theta) + B(\theta, S_A) \nabla S_A^T \cdot (\mathbf{K} \nabla S_A) \\ 654 &+ C(\theta, S_A) [\nabla S_A^T \cdot (\mathbf{K} \nabla \theta) + \nabla \theta^T \cdot (\mathbf{K} \nabla S_A)], \end{aligned} \quad (101)$$

where A , B , and C are all functions of θ and S alone given by

$$\begin{aligned} A(\theta, S_A) &= \frac{1}{c_p^R} \frac{\partial c_p^R}{\partial \theta}, \quad B(\theta, S_A) = \frac{1}{c_p^R} \left(\frac{\partial \mu_R}{\partial S_A} - \theta \frac{\partial^2 \mu_R}{\partial S_A \partial \theta} \right), \\ 655 & \\ C(\theta, S_A) &= \frac{1}{c_p^R} \frac{\partial c_p^R}{\partial S_A} = -\frac{\theta}{c_p^R} \frac{\partial^2 \mu_R}{\partial \theta^2}. \end{aligned} \quad (102)$$

656 Given that the modifications to the potential temperature equation detailed
 657 above should be strictly equivalent to using the (Θ, S_A) formulation proposed

658 by IOC et al. (2010), the actual benefits of switching to Conservative Tem-
 659 perature are no longer obvious. From a computational viewpoint, adding the
 660 nonconservative term (101) and modifying the current boundary condition
 661 as per (92) would be straightforward. Moreover, since OGCMs estimate $\nabla\theta$,
 662 ∇S_A , $\mathbf{K}\nabla\theta$ and $\mathbf{K}\nabla S_A$ as part of computing the heat and salt fluxes, diag-
 663 nosing $\dot{\theta}_{irr}$ would come at little additional cost, as to do so would only require
 664 additional routines for the 4 functions of θ and S_A alone, namely c_p^R , A , B ,
 665 and C . As regards to estimating the poleward heat transport, it could easily
 666 be diagnosed in terms of CT as recommended by McDougall (2003), since
 667 Θ can be diagnosed from (θ, S_A) (using the routine `gsw_CT_from_pt.m`
 668 from the GSW Library for instance). Since a (θ, S_A) formulation can be con-
 669 structed that is strictly equivalent to the (Θ, S_A) formulation proposed by
 670 IOC et al. (2010), it follows that the decision to switch to CT should be
 671 motivated on a careful evaluation of the computational and physical advan-
 672 tages of each formulation. Note, however, that even though the (θ, S_A) and
 673 (Θ, S_A) formulations discussed in this section are constructed to be strictly
 674 equivalent mathematically, it would be of interest to test whether this is also
 675 the case at the discretised level, as differences in results could shed light on
 676 whether the budget of derived nonconservative quantities such as buoyancy
 677 or entropy can be expected to be accurately represented in numerical ocean
 678 models.

679 **Acknowledgements**

680 Numerous discussions with Trevor McDougall over the past few years
 681 greatly helped in defining and clarifying the issues addressed in this paper,
 682 and contributed to point the way to the results of the last section. Con-

683 structive criticism by three anonymous referees greatly helped clarify and
684 improve the paper. This project was supported by the NERC funded IN-
685 SPECT project NE/K016083/1.

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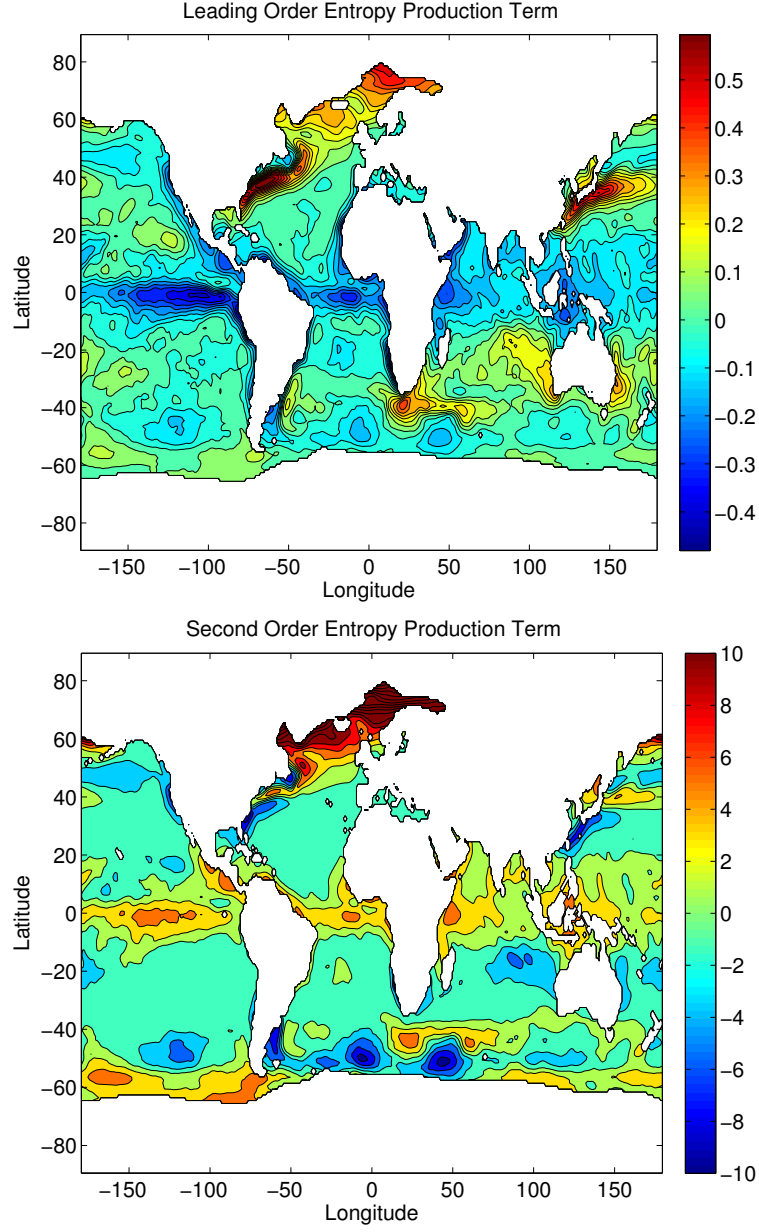


Figure 2: Leading order entropy production term $-\tilde{Q}/\bar{T}$ (top panel, in W.m⁻².K⁻¹) and second order entropy production term $-\tilde{Q}(1/T - 1/\bar{T})$ (bottom panel, in mW.m⁻².K⁻¹)

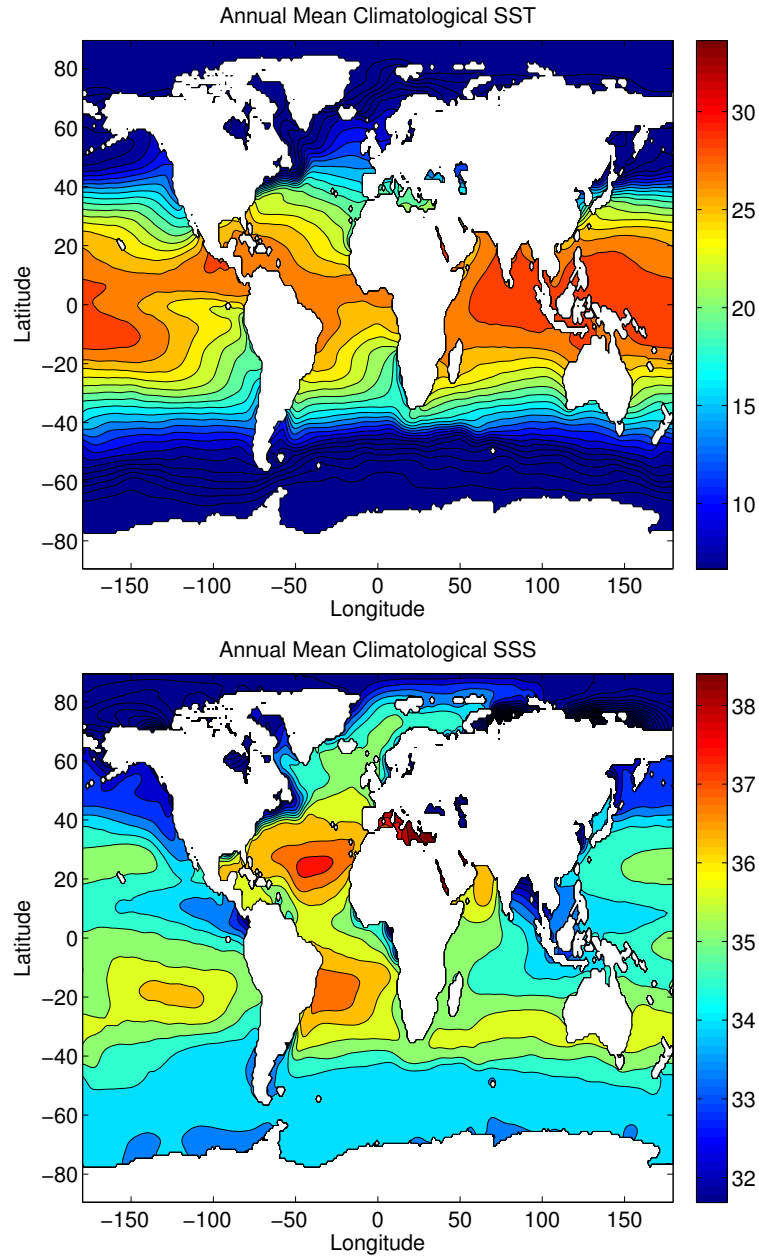


Figure 3: Climatological annual mean sea surface temperature (top panel, in degrees Celsius) and sea surface salinity (bottom panel, in g/kg) from the World Ocean Database 2013.

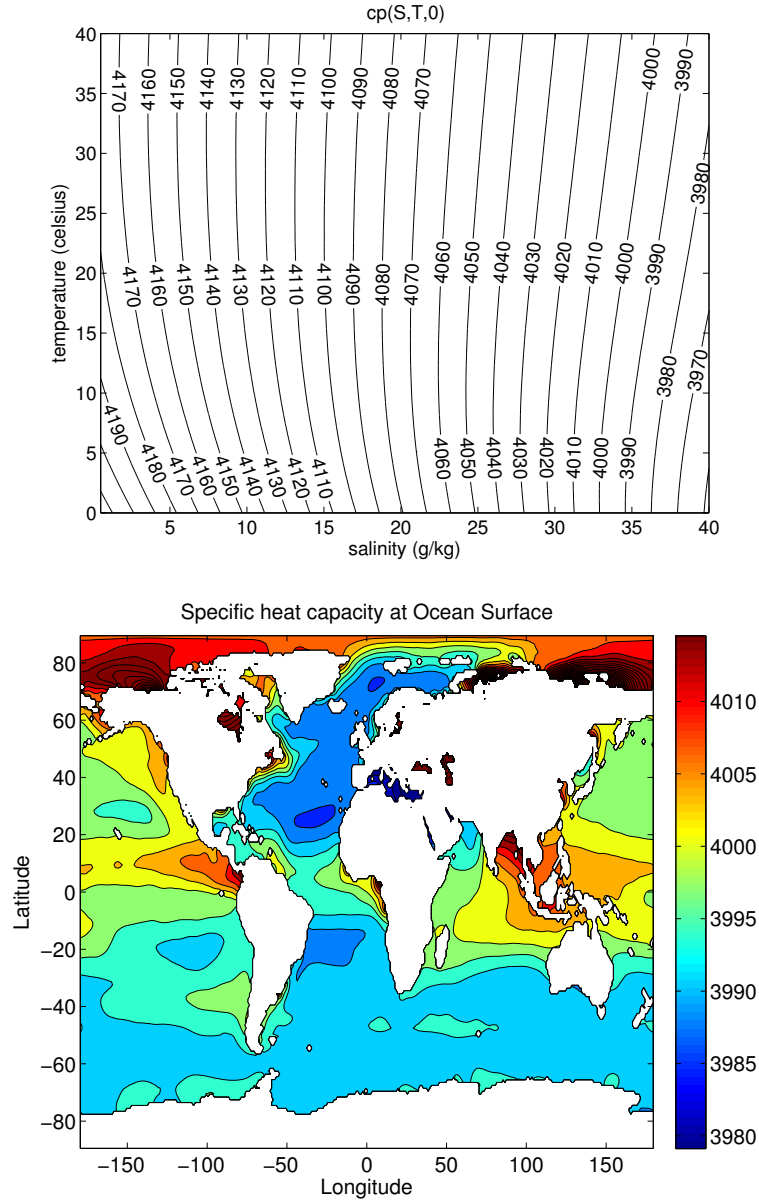


Figure 4: (Top panel) Specific heat capacity c_p (in $\text{J.K}^{-1}.\text{kg}^{-1}$) as a function of temperature and salinity at mean atmospheric pressure, illustrating the strong dependence of c_p upon salinity. (Bottom panel) Surface distribution of c_p (same units) for the climatological annual mean temperature and salinity fields depicted in Fig. 3.

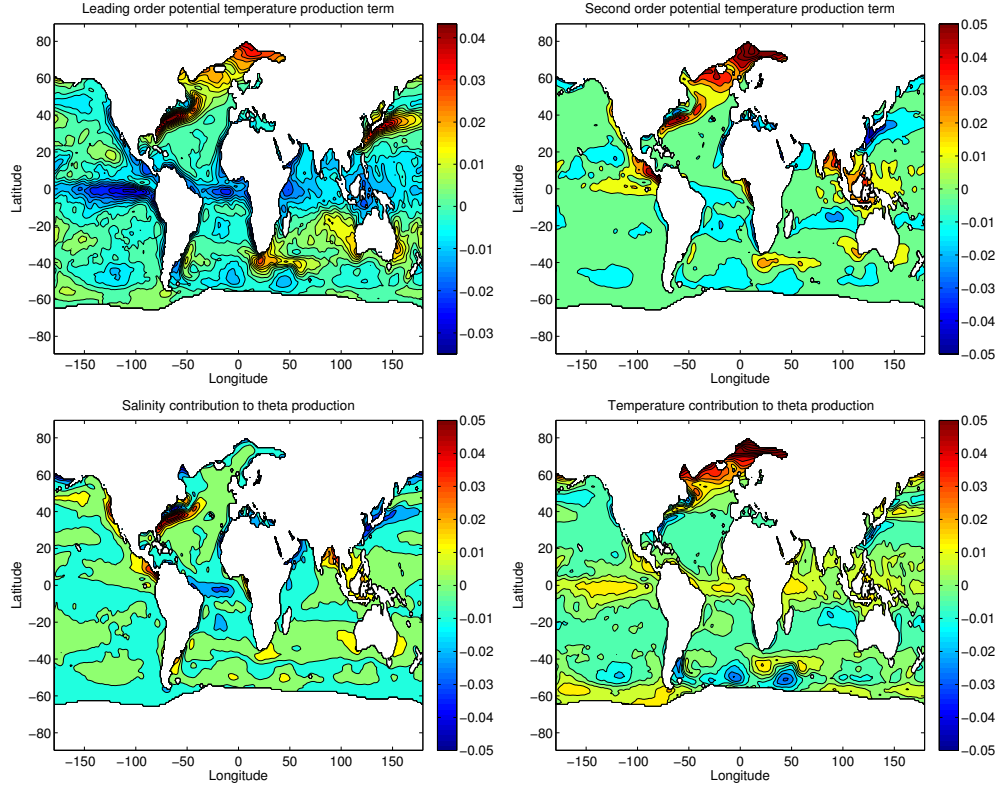


Figure 5: Leading order production term $-\tilde{Q}/\bar{c}_p^R$ (top left panel, in $\text{kg.m}^{-2}.\text{s}^{-1}.\text{K}$) and second order production term $-\tilde{Q}/(1/c_p^R - 1/\bar{c}_p^R)$ (top right panel, in $10^{-3}\text{kg.m}^{-2}.\text{s}^{-1}.\text{K}$). Decomposition of the top right panel into a salinity anomaly contribution (bottom left panel) and temperature anomaly contribution (bottom right panel), in same units as top right panel.

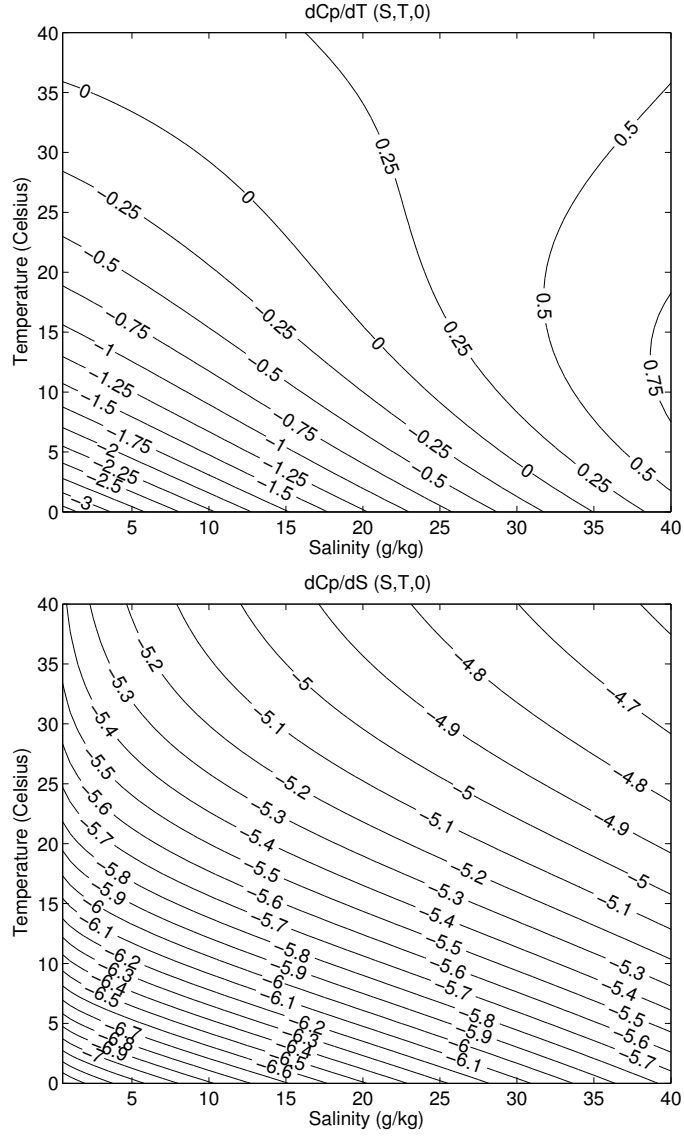


Figure 6: Derivative of the specific heat capacity with respect to temperature (top panel) and to salinity (bottom panel) as a function of temperature and salinity at mean atmospheric surface pressure. Units are respectively $J.K^{-2}.kg^{-1}$ and $J.K^{-1}.kg^{-1}.(g/kg)^{-1}$.