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Ω-results for Beurling's zeta function and lower bounds for the generalised Dirichlet divisor problem¹

Titus W. Hilberdink

Abstract

In this paper we study generalised prime systems for which the integer counting function $N_{\mathcal{P}}(x)$ is asymptotically well-behaved, in the sense that $N_{\mathcal{P}}(x) = \rho x + O(x^\beta)$, where ρ is a positive constant and $\beta < \frac{1}{2}$. For such systems, the associated zeta function $\zeta_{\mathcal{P}}(s)$ is holomorphic for $\sigma = \Re s > \beta$. We prove that for $\beta < \sigma < \frac{1}{2}$, $\int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt = \Omega(T^{2-2\sigma-\varepsilon})$ for any $\varepsilon > 0$, and also for $\varepsilon = 0$ for all such σ except possibly one value.

The Dirichlet divisor problem for generalised integers concerns the size of the error term in $N_{k\mathcal{P}}(x) - \text{Res}_{s=1}(\zeta_{\mathcal{P}}(s)^k x^s/s)$, which is $O(x^\theta)$ for some $\theta < 1$. Letting α_k denote the infimum of such θ , we show that $\alpha_k \geq \frac{1}{2} - \frac{1}{2k}$.

Keywords: Beurling's generalised primes, Dirichlet divisor problem.

AMS Mathematics subject classification 2010: 11N80.

1. Introduction

A *generalised prime system* (or *g-prime system*) $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, p_3, \dots$ satisfying

$$1 < p_1 \leq p_2 \leq \dots \leq p_n \leq \dots$$

and for which $p_n \rightarrow \infty$ as $n \rightarrow \infty$. From these can be formed the system $\mathcal{N}$ of *generalised integers* or *Beurling integers*; that is, the numbers of the form

$$p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$$

where $k \in \mathbb{N}$ and $a_1, \dots, a_k \in \mathbb{N}_0$.² Such systems were first introduced by Beurling [2] and have been studied by many authors since then (see in particular [1]). Define the g-integer counting function $N_{\mathcal{P}}(x)$ and the associated Beurling zeta function, respectively, by

$$N_{\mathcal{P}}(x) = \sum_{n \in \mathcal{N}, n \leq x} 1, \quad \zeta_{\mathcal{P}}(s) = \sum_{n \in \mathcal{N}} \frac{1}{n^s}.$$

(Here, $\sum_{n \in \mathcal{N}}$ means a sum over all the g-integers, counting multiplicities.) In this paper, we shall be concerned with g-prime systems for which

$$N_{\mathcal{P}}(x) = \rho x + O(x^\beta), \tag{1.1}$$

for some $\beta < \frac{1}{2}$ and $\rho > 0$. Then $\zeta_{\mathcal{P}}(s)$ is defined and holomorphic for $\Re s > 1$, and has an analytic continuation to the half-plane $\Re s > \beta$ except for a simple pole at $s = 1$ with residue ρ . Furthermore, $\zeta_{\mathcal{P}}(s)$ has *finite order* for $\Re s > \beta$; i.e. $\zeta_{\mathcal{P}}(\sigma + it) = O(|t|^\lambda)$ for some λ for $\sigma > \beta$. Let $\mu_{\mathcal{P}}(\sigma)$ denote the infimum of all such λ . It is well-known that $\mu_{\mathcal{P}}(\sigma)$ is non-negative,

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²Here, $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{P} = \{2, 3, 5, \dots\}$ — the set of primes.

decreasing, and convex (and hence continuous) (see, for example, [5]). For $\mathcal{P} = \mathbb{P}$ (so that $\mathcal{N} = \mathbb{N}$), the Lindelöf Hypothesis is the conjecture that $\mu_{\mathbb{P}}(\sigma) = \mu_0(\sigma)$ for all σ , where

$$\mu_0(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \\ 0 & \text{if } \sigma \geq \frac{1}{2} \end{cases}.$$

In [4], it was proven that for all g-prime systems satisfying (1.1), $\mu_{\mathcal{P}}(\sigma)$ must be *at least* as large as $\mu_0(\sigma)$: i.e. $\mu_{\mathcal{P}}(\sigma) \geq \frac{1}{2} - \sigma$ for $\sigma \in (\beta, \frac{1}{2})$. In this paper we prove a stronger result by considering the mean square behaviour of $\zeta_{\mathcal{P}}(\sigma + it)$. For $\sigma > \beta$, define $\nu_{\mathcal{P}}(\sigma)$ to be the infimum of numbers λ such that

$$\int_1^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt = O(T^{1+2\lambda}).$$

As in the case of $\mu_{\mathcal{P}}(\sigma)$, $\nu_{\mathcal{P}}(\sigma)$ is non-negative and convex decreasing (cf. [6], §7.8). Trivially, $\nu_{\mathcal{P}}(\sigma) \leq \mu_{\mathcal{P}}(\sigma)$. We show here that $\nu_{\mathcal{P}}(\sigma) \geq \mu_0(\sigma)$. In fact we prove slightly more.

Theorem 1

Let $\mathcal{P}$ be a g-prime system for which (1.1) holds for some $\beta < \frac{1}{2}$ and $\rho > 0$. Then $\nu_{\mathcal{P}}(\sigma) \geq \mu_0(\sigma)$ for $\sigma \in (\beta, \frac{1}{2})$. Furthermore,

$$\int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt = o(T^{2-2\sigma}) \quad (1.2)$$

can hold for at most one value of σ in this range. In this case $T^{2\sigma-2} \int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt$ is unbounded for all other values of σ .

Remark. For $\mathcal{P} = \mathbb{P}$, we have $\nu_{\mathcal{P}}(\sigma) = \mu_0(\sigma)$, which shows the first part of Theorem 1 is best possible. However, in this case we have the asymptotic formula

$$\int_1^T |\zeta(\sigma + it)|^2 dt \sim \frac{\zeta(2-2\sigma)}{(2\pi)^{1-2\sigma}(2-2\sigma)} T^{2-2\sigma}$$

for $0 < \sigma < \frac{1}{2}$, showing that the exceptional value need not exist. In fact it seems unlikely an exceptional value exists and hence that $\int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt = \Omega(T^{2-2\sigma})$ for all $\sigma \in (\beta, \frac{1}{2})$, but we cannot quite show this. Furthermore it seems plausible that we should have $\int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt \geq C_{\sigma} T^{2-2\sigma}$ for some $C_{\sigma} > 0$.

2. Dirichlet divisor problems for g-primes

For a g-prime system satisfying (1.1) (with $\beta < 1$), we can study the equivalent of the Dirichlet divisor problem concerning the error term in the asymptotic formula for the average of the ‘generalised divisor’ function. For $k \in \mathbb{N}$, let $k\mathcal{P}$ denote the g-prime system obtained from $\mathcal{P}$ by letting every g-prime from $\mathcal{P}$ be counted k times. (If an original g-prime has multiplicity m , then in the new system it will have multiplicity km .) The Beurling zeta function of $k\mathcal{P}$ is

$$\zeta_{k\mathcal{P}}(s) = \zeta_{\mathcal{P}}(s)^k.$$

By standard methods using Perron’s formula,

$$N_{k\mathcal{P}}(x) = \text{Res}_{s=1} \left\{ \frac{\zeta_{\mathcal{P}}(s)^k}{s} x^s \right\} + \Delta_{\mathcal{P},k}(x) = xP_{k-1}(\log x) + \Delta_{\mathcal{P},k}(x),$$

where $P_{k-1}(\cdot)$ is a polynomial of degree $k-1$ and $\Delta_{\mathcal{P},k}(x) = O(x^\theta)$ for some $\theta < 1$, depending on k . Let α_k denote the infimum of such θ . The *generalised Dirichlet divisor problem* is the problem of determining α_k . Also let β_k denote the infimum of ϕ for which

$$\int_0^x \Delta_{\mathcal{P},k}(y)^2 dy = O(x^{1+2\phi}).$$

Trivially, $\beta_k \leq \alpha_k$.

For $\mathbb{P}$, it is known that

$$\alpha_k \geq \beta_k \geq \frac{1}{2} - \frac{1}{2k} \quad (2.1)$$

and it is conjectured that there is equality throughout (actually $\beta_k = \frac{1}{2} - \frac{1}{2k}$ for all k is equivalent to the Lindelöf Hypothesis — see [6], Theorem 13.4). We use Theorem 1 to show that (2.1) remains true for $\mathcal{P}$ satisfying (1.1). In fact we have the following two corollaries:

Corollary 2

Let $\mathcal{P}$ satisfy (1.1) for some $\beta < \frac{1}{2}$. Then for $\sigma \in (\beta, \frac{1}{2} - \frac{1}{2k})$,

$$\int_{-\infty}^{\infty} \frac{|\zeta_{\mathcal{P}}(\sigma + it)|^{2k}}{|\sigma + it|^2} dt \quad (2.2)$$

diverges. Further, if $\frac{1}{2} - \frac{1}{2k}$ is not the exceptional value in (1.2), then the integral also diverges for $\sigma = \frac{1}{2} - \frac{1}{2k}$.

Corollary 3

Let $\mathcal{P}$ satisfy (1.1) for some $\beta < \frac{1}{2}$. With α_k and β_k as above, $\alpha_k \geq \beta_k \geq \max\{\beta, \frac{1}{2} - \frac{1}{2k}\}$.

3. Proofs

Proof of Theorem 1. If $\nu_{\mathcal{P}}(\sigma') < \frac{1}{2} - \sigma'$ for some $\sigma' \in (\beta, \frac{1}{2})$ then, by continuity of $\nu_{\mathcal{P}}(\cdot)$, $\nu_{\mathcal{P}}(\sigma) < \frac{1}{2} - \sigma$ throughout some interval around σ' and (1.2) holds for all such σ ; in particular for two such values. We shall show that this is impossible.

Suppose, for a contradiction, that (1.2) holds for $\sigma = \sigma_0, \sigma_1$ where $\beta < \sigma_0 < \sigma_1 < \frac{1}{2}$.

For $N \geq 1$ let $\zeta_{N,\mathcal{P}}(s) = \sum_{n \leq N} n^{-s}$, where the sum ranges over $n \in \mathcal{N}$. As was stated in [4] (and shown in [3]), for $\sigma < \frac{1}{2}$ there exist constants $c_1, c_2 > 0$ such that for $R \geq c_1 N$,

$$\sum_{r=1}^R \int_0^{2r-1} |\zeta_{N,\mathcal{P}}(\sigma + it)|^2 dt \geq c_2 R^2 N^{1-2\sigma}. \quad (3.1)$$

Also, writing $s = \sigma + it$, and following the arguments in [3], we have

$$\zeta_{N,\mathcal{P}}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_{\mathcal{P}}(s+w)N^w}{w} dw + O\left(\frac{N^c}{T(c+\sigma-1)}\right) + O\left(\frac{N^{1-\sigma}}{T} \sum_{\substack{\frac{N}{2} < n < 2N \\ n \in \mathcal{N}}} \frac{1}{|n-N|}\right), \quad (3.2)$$

for $|t| < T$, $c > 1 - \sigma$ and $N \notin \mathcal{N}$. We shall put $c = 1 - \sigma + \frac{1}{\log N}$ and choose N in such a way that $(N - \alpha, N + \alpha) \cap \mathcal{N} = \emptyset$. (As was shown in [4], this is possible for arbitrarily large N if

$0 < \alpha < \frac{1}{4\rho}$.) With this choice of N , the final sum in (3.2) was shown to be $O(\sqrt{N})$. As such (3.2) becomes

$$\zeta_{N,\mathcal{P}}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_{\mathcal{P}}(s+w)N^w}{w} dw + O\left(\frac{N^{\frac{3}{2}-\sigma}}{T}\right). \quad (3.3)$$

Now put $\sigma = \sigma_1$ and push the contour in the integral to the left as far as $\Re w = \sigma_0 - \sigma_1 < 0$, picking up the residues at $w = 0$ and $w = 1 - s$ (since $|t| < T$).

The contribution along the horizontal line $[\sigma_0 - \sigma_1 + iT, c + iT]$ is, in modulus, less than

$$\frac{1}{2\pi T} \int_{\sigma_0 - \sigma_1}^c N^y |\zeta_{\mathcal{P}}(\sigma_1 + y + i(t+T))| dy.$$

Using the uniform bound $|\zeta_{\mathcal{P}}(\sigma + it)| = O(t^{\frac{1-\sigma}{1-\beta}+\varepsilon})$, this is at most a constant times

$$\frac{1}{T} \int_{\sigma_0 - \sigma_1}^{1-\sigma_1} T^{\frac{1-\sigma_1-y}{1-\beta}+\varepsilon} N^y dy + \frac{1}{T} \int_{1-\sigma_1}^{1-\sigma_1+\frac{1}{\log N}} T^\varepsilon N^y dy = O(T^{\frac{\beta-\sigma_0}{1-\beta}+\varepsilon} N^{\sigma_0-\sigma_1}) + O(T^{\varepsilon-1} N^{1-\sigma_1}). \quad (3.4)$$

Similarly on $[\sigma_0 - \sigma_1 - iT, c - iT]$.

The integral along $\Re w = \sigma_0 - \sigma_1$ is at most

$$\begin{aligned} \frac{N^{\sigma_0-\sigma_1}}{2\pi} \int_{-T}^T \frac{|\zeta_{\mathcal{P}}(\sigma_0 + i(t+y))|}{\sqrt{(\sigma_1 - \sigma_0)^2 + y^2}} dy &= O\left(N^{\sigma_0-\sigma_1} \int_1^{2T} \frac{|\zeta_{\mathcal{P}}(\sigma_0 + iy)|}{y} dy\right) \\ &= o(N^{\sigma_0-\sigma_1} T^{\frac{1}{2}-\sigma_0}), \end{aligned} \quad (3.5)$$

using³ the hypothetical bound $\int_0^T |\zeta_{\mathcal{P}}(\sigma_0 + it)|^2 dt = o(T^{2-2\sigma_0})$.

The residues at $w = 0$ and $w = 1 - s$ are, respectively, $\zeta_{\mathcal{P}}(s)$ and $\rho N^{1-s}/(1-s) = O(\frac{N^{1-\sigma_1}}{|t|+1})$. Putting (3.3), (3.4), and (3.5) together gives

$$\zeta_{N,\mathcal{P}}(\sigma_1 + it) = \zeta_{\mathcal{P}}(\sigma_1 + it) + O\left(\frac{N^{1-\sigma_1}}{|t|+1}\right) + O(N^{1-\sigma_1} T^{\varepsilon-1}) + o(N^{\sigma_0-\sigma_1} T^{\frac{1}{2}-\sigma_0}) + O\left(\frac{N^{\frac{3}{2}-\sigma_1}}{T}\right),$$

for $|t| < T$. (Note that the first O -term in (3.4) is superfluous since $\frac{\beta-\sigma_0}{1-\beta} < \frac{1}{2} - \sigma_0$.) Hence, using $(a+b+c+d+e)^2 \leq 5(a^2+b^2+c^2+d^2+e^2)$, we have

$$|\zeta_{N,\mathcal{P}}(\sigma_1 + it)|^2 \leq 5|\zeta_{\mathcal{P}}(\sigma_1 + it)|^2 + O\left(\frac{N^{2-2\sigma_1}}{t^2+1}\right) + O(N^{2-2\sigma_1} T^{2\varepsilon-2}) + o(N^{2\sigma_0-2\sigma_1} T^{1-2\sigma_0}) + O\left(\frac{N^{3-2\sigma_1}}{T^2}\right).$$

Now apply $\sum_{r=1}^R \int_0^{2^{r-1}} \dots dt$ to both sides to give (for $2R-1 < T$)

$$\begin{aligned} \sum_{r=1}^R \int_0^{2^{r-1}} |\zeta_{N,\mathcal{P}}(\sigma_1 + it)|^2 dt &= O\left(\sum_{r=1}^R \int_0^{2^{r-1}} |\zeta_{\mathcal{P}}(\sigma_1 + it)|^2 dt\right) + O\left(\sum_{r=1}^R \int_0^{2^{r-1}} \frac{N^{2-2\sigma_1}}{(t+1)^2} dt\right) \\ &\quad + O(R^2 N^{2-2\sigma_1} T^{2\varepsilon-2}) + O\left(\frac{R^2 N^{3-2\sigma_1}}{T^2}\right) + o(R^2 N^{2(\sigma_0-\sigma_1)} T^{1-2\sigma_0}) \\ &= o(R^{3-2\sigma_1}) + O(RN^{2-2\sigma_1}) + O(R^2 N^{2-2\sigma_1} T^{2\varepsilon-2}) + O\left(\frac{R^2 N^{3-2\sigma_1}}{T^2}\right) + o(R^2 N^{2(\sigma_0-\sigma_1)} T^{1-2\sigma_0}) \end{aligned}$$

³If $f \geq 0$ and $\int_0^T f^2 = o(T^\lambda)$ (some $\lambda > 1$), then $\int_{T/2}^T \frac{f(y)}{y} dy \leq \frac{2}{T} \int_0^T f \leq \frac{2}{T} \sqrt{T \int_0^T f^2} = o(T^{\frac{\lambda-1}{2}})$, and $\int_1^T \frac{f(y)}{y} dy = o(T^{\frac{\lambda-1}{2}})$ follows.

using (1.2) for σ_1 . Let $T = 2R$. The left-hand side above is at least $c_2 R^2 N^{1-2\sigma_1}$ by (3.1) if $R \geq c_1 N$. Dividing both sides through by $R^2 N^{1-2\sigma_1}$ gives

$$c_2 \leq o\left(\left(\frac{R}{N}\right)^{1-2\sigma_1}\right) + O\left(\frac{N}{R}\right) + O(NR^{2\varepsilon-2}) + O\left(\frac{N^2}{R^2}\right) + o\left(\left(\frac{R}{N}\right)^{1-2\sigma_0}\right). \quad (3.6)$$

Put $R = KN$ where $K \geq c_1$ is a fixed, but arbitrary, constant. Letting $N \rightarrow \infty$, the o -terms both tend to zero as does the middle O -term. Hence

$$c_2 \leq \frac{A}{K} + \frac{B}{K^2}$$

for some absolute constants A, B . But K can be made arbitrarily large, so this gives a contradiction.

For the final part, suppose (1.2) holds for $\sigma = \sigma_0$ say. If $\int_0^T |\zeta_{\mathcal{P}}(\sigma' + it)|^2 dt = O(T^{2-2\sigma'})$ for some $\sigma' \in (\beta, \frac{1}{2})$ with $\sigma' \neq \sigma_0$, then (1.2) actually holds for all σ between σ_0 and σ' . (This follows from the Phragmen-Lindelöf Theorem for a strip (see [6], §7.8, with ε in the place of C)). This was shown to be impossible, and hence $T^{2\sigma-2} \int_0^T |\zeta_{\mathcal{P}}(\sigma' + it)|^2 dt$ must be unbounded for all $\sigma \neq \sigma_0$. □

Now we apply Theorem 1 to find lower bounds in the Dirichlet divisor problem. Note that Theorem 1 actually shows that given $\varepsilon > 0$,

$$\int_{T/2}^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt = \Omega(T^{2-2\sigma-\varepsilon}),$$

for if it was $o(T^{2-2\sigma-\varepsilon})$, then by telescoping it would follow that $\int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt = o(T^{2-2\sigma-\varepsilon})$ which is false.

Proofs of Corollaries 2 and 3. By Hölder's inequality,

$$\int_{T/2}^T |\zeta_{\mathcal{P}}(\sigma + it)|^{2k} dt \geq \frac{2^{k-1}}{T^{k-1}} \left(\int_{T/2}^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt \right)^k,$$

for every $k \in \mathbb{N}$. By Theorem 1, given $\varepsilon > 0$, $\int_{T/2}^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 dt \geq aT^{2-2\sigma-\varepsilon}$ for some $a > 0$ and some arbitrarily large T . Hence for such T ,

$$\int_{T/2}^T |\zeta_{\mathcal{P}}(\sigma + it)|^{2k} dt \geq a^k T^{k(1-2\sigma)+1-\varepsilon k}.$$

It follows that

$$\int_{T/2}^T \frac{|\zeta_{\mathcal{P}}(\sigma + it)|^{2k}}{|\sigma + it|^2} dt \geq a' T^{k(1-2\sigma)-1-\varepsilon k}$$

for some $a' > 0$. But for $\sigma < \frac{1}{2} - \frac{1}{2k}$, we have $k(1-2\sigma) - 1 > 0$. Hence for ε sufficiently small, $k(1-2\sigma) - 1 - \varepsilon k > 0$ also, and so $\int_{T/2}^T \frac{|\zeta_{\mathcal{P}}(\sigma + it)|^{2k}}{|\sigma + it|^2} dt \not\rightarrow 0$ as $T \rightarrow \infty$, and Corollary 2 follows. Of course, if $\frac{1}{2} - \frac{1}{2k}$ is not the exceptional value in Theorem 1, then we can take $\varepsilon = 0$ in the above and the result also holds for $\sigma = \frac{1}{2} - \frac{1}{2k}$.

Let γ_k be the infimum of σ (with $\sigma > \beta$) for which $\int_{-\infty}^{\infty} \frac{|\zeta_{\mathcal{P}}(\sigma+it)|^{2k}}{|\sigma+it|^2} dt$ converges. By Corollary 2, $\gamma_k \geq \frac{1}{2} - \frac{1}{2k}$. An identical argument as in the $\mathcal{P} = \mathbb{P}$ case (see [6], Theorem 12.5) shows that $\gamma_k = \beta_k$. (The argument is simply based upon Parseval's formula for Mellin transforms, which in this case is the identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta_{\mathcal{P}}(\sigma+it)|^{2k}}{|\sigma+it|^2} dt = \int_0^{\infty} \frac{\Delta_{\mathcal{P},k}(x)^2}{x^{1+2\sigma}} dx$$

for σ in some interval $(\theta, 1)$ with $\theta < 1$.) Hence $\beta_k \geq \frac{1}{2} - \frac{1}{2k}$. □

4. On the line $\sigma = \frac{1}{2}$

In this article, we have considered the mean-value along vertical lines $\Re s = \sigma$ with $\sigma < \frac{1}{2}$. This raises the question of what happens on the line $\sigma = \frac{1}{2}$. For $\mathcal{P} = \mathbb{P}$, we have $\int_0^T |\zeta(\frac{1}{2} + it)|^2 dt \sim T \log T$, so do we have $\int_0^T |\zeta_{\mathcal{P}}(\frac{1}{2} + it)|^2 dt = \Omega(T \log T)$ in general? As in the $\sigma < \frac{1}{2}$ case, we relate the behaviour of the mean-square value at $\sigma = \frac{1}{2}$ to the behaviour of the mean-square for some $\sigma = \sigma_0 < \frac{1}{2}$.

Theorem 4

Let $\mathcal{P}$ be a g -prime system for which (1.1) holds. If $\int_1^T \frac{|\zeta_{\mathcal{P}}(\sigma+it)|}{t} dt = o((T \log T)^{\frac{1}{2}-\sigma})$ for some $\sigma \in (\beta, \frac{1}{2})$, then $\int_0^T |\zeta_{\mathcal{P}}(\frac{1}{2} + it)|^2 dt = \Omega(T \log T)$.

Note that the assumption is implied by $\int_1^T |\zeta_{\mathcal{P}}(\sigma+it)|^2 dt = o(T^{2-2\sigma}(\log T)^{1-2\sigma})$.

Sketch of Proof. We follow the proof of Theorem 1 as much as possible, this time taking $\sigma_1 = \frac{1}{2}$.

Using the argument in [3] for $\sigma = \frac{1}{2}$, (3.1) becomes: *there exist constants $c_1, c_2 > 0$ such that for $R \geq c_1 N / \log N$,*

$$\sum_{r=1}^R \int_0^{2r-1} \left| \zeta_{N,\mathcal{P}}\left(\frac{1}{2} + it\right) \right|^2 dt \geq c_2 R^2 \log N. \quad (4.1)$$

To see this, note that we have

$$\int_0^T \left| \zeta_{N,\mathcal{P}}\left(\frac{1}{2} + it\right) \right|^2 dt = T \sum_{n \leq N}^* \frac{1}{n} + 2 \sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{m < n} \frac{S_{m,n}(T)}{\sqrt{m}},$$

where $S_{m,n}(T) = \frac{\sin(T \log(n/m))}{\log(n/m)}$. (Here $m, n \in \mathcal{N}$ and the $*$ indicates that any multiplicities must be squared.) In any case, we have $\sum_{n \leq N}^* \frac{1}{n} \geq \sum_{n \leq N} \frac{1}{n} \geq k_1 \log N$ for some $k_1 > 0$.⁴ For $m \leq \frac{n}{2}$, $|S_{m,n}(T)| \leq 1/\log 2$, so this part of the double sum is $O(\sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{m \leq n/2} \frac{1}{\sqrt{m}}) = O(N)$. Thus, for some positive constants k_1, k_2 , independent of T and N ,

$$\int_0^T \left| \zeta_{N,\mathcal{P}}\left(\frac{1}{2} + it\right) \right|^2 dt \geq k_1 T \log N + 2 \sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{\frac{n}{2} < m < n} \frac{S_{m,n}(T)}{\sqrt{m}} - k_2 N.$$

Putting $T = 2r - 1$ for $r = 1, 2, \dots, R$, and summing both sides gives, on noticing that $\sum_{r=1}^R \sin((2r-1) \log \frac{n}{m}) = \frac{\sin^2(R \log n/m)}{\sin(\log n/m)} \geq 0$ since $0 < \log n/m < \log 2$,

$$\sum_{r=1}^R \int_0^{2r-1} \left| \zeta_{N,\mathcal{P}}\left(\frac{1}{2} + it\right) \right|^2 dt \geq k_1 R^2 \log N - k_2 RN,$$

⁴This follows readily from $N_{\mathcal{P}}(x) \sim \rho x$.

and (4.1) follows.

In (3.2), we need a better estimate for the final sum. Let $M \in \mathbb{N}$. Then, with N such that $(N - \alpha, N + \alpha) \cap \mathcal{N} = \emptyset$,

$$\begin{aligned} \sum_{\substack{\frac{N}{2} < n < 2N \\ n \in \mathcal{N}}} \frac{1}{|n - N|} &= \sum_{m=1}^M \sum_{\alpha N^{\frac{m-1}{M}} \leq |n-N| < \alpha N^{\frac{m}{M}}} \frac{1}{|n - N|} + O(1) \\ &\leq \frac{1}{\alpha} \sum_{m=1}^M \frac{1}{N^{\frac{m-1}{M}}} \left(N(N + \alpha N^{m/M}) - N(N - \alpha N^{m/M}) \right) + O(1) \\ &= O(N^{1/M}) + O(N^\beta), \end{aligned}$$

using (1.1). Since M is arbitrary, this is $O(N^{\beta+\varepsilon})$ for every $\varepsilon > 0$ in any case. Thus (3.3) becomes

$$\zeta_{N,\mathcal{P}}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_{\mathcal{P}}(s+w)N^w}{w} dw + O\left(\frac{N^{\frac{1}{2}+\beta+\varepsilon}}{T}\right).$$

The analysis up to (3.5) remains the same (with $\sigma_0 = \sigma$ and $\sigma_1 = \frac{1}{2}$) but in (3.5) we use the bound assumed in the statement to give $o(N^{\sigma-\frac{1}{2}}(T \log T)^{\frac{1}{2}-\sigma})$. The arguments following (3.5) remain valid and we put $T = 2R$ again, but this time we divide through by $R^2 \log N$. On assuming $\int_0^T |\zeta_{\mathcal{P}}(\frac{1}{2} + it)|^2 dt = o(T \log T)$, (3.6) now becomes

$$c_2 \leq o\left(\frac{\log R}{\log N}\right) + O\left(\frac{N}{R \log N}\right) + O\left(\frac{NR^{2\varepsilon-2}}{\log N}\right) + O\left(\frac{N^{1+2\beta+2\varepsilon}}{R^2}\right) + o\left(\left(\frac{R \log R}{N}\right)^{1-2\sigma} \frac{1}{\log N}\right).$$

Put $R = KN / \log N$ where $K \geq c_1$ is a fixed, but arbitrary, constant. Letting $N \rightarrow \infty$, all the terms tend to zero except the first O -term. Hence

$$c_2 \leq \frac{A}{K}$$

for some absolute constant A . As K can be made arbitrarily large, this gives a contradiction. Hence $\int_0^T |\zeta_{\mathcal{P}}(\frac{1}{2} + it)|^2 dt = \Omega(T \log T)$. □

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