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# Estimating Systemic Risk Using Composite Quantile Regression

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## Abstract

Value at Risk (VaR) and Average Value at Risk (AVaR) are among the most widely-used risk measures by market participants to assess the risk of individual financial firms and institutions. Despite their popularity, both measures fail to account for spillover effects between firms. To address this limitation, the CoVaR (Conditional Value at Risk) measure was introduced, which defines the VaR of a financial system conditional on the state of another institution. The traditional approach to estimating CoVaR involves a regression model combined with a quantile method to estimate the model's parameters. This paper proposes a composite quantile regression method to enhance the accuracy of CoVaR estimation. We apply this methodology to several U.S. companies across various sectors, including finance, consumer goods, energy, industry, and technology. An analysis of the out-of-sample forecast accuracy using two popular backtesting criteria demonstrates that the composite quantile method provides more accurate CoVaR estimates than the standard quantile method. All computation codes are freely available in both R and MATLAB.

**Keywords** Composite quantile regression method · CoVaR systemic risk · Risk map · Violations

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## 1 Introduction

In recent years, much attention has been devoted to enhancing risk measurement tools to protect a firm or investment portfolio against unexpected events. Among these tools, Value at Risk (VaR) has emerged as one of the most widely used and established risk measures (e.g., Markowitz 1952, Roy 1952, and Bernard et al., 2023). This measure estimates the financial risk of an asset, portfolio, or investment over a specified period with a given confidence level. Several reasons lead VaR to be welcomed by investors; on top of them, one can mention its minor computation costs. However, despite its advantages, this measure also has some undeniable drawbacks. VaR is not a coherent risk measure and performs poorly in assessing risk in the loss distribution's fat tails (e.g., Cruz et al. 2015 and Emmer et al. 2015). The Average Value at Risk (AVaR) risk measure is introduced by Rockafellar et al. (2000) to overcome these weak points. AVaR is coherent and consistent with the fat tail loss distributions (McNeil et al., 2015). While AVaR performance is better than VaR in considering the risk of fat tails, this measure also suffers from more computation costs than VaR. Both VaR and AVaR have a significant drawback that makes them unsuitable for comprehensive risk measurement: they do not account for risk spillover effects among institutions. Despite being the two most widely used risk measures in financial markets, this limitation undermines their effectiveness in accurately assessing systemic risk.

It is proven that, especially during financial crises, losses in some institutions tend to spread across other institutions and threaten the whole system. This phenomenon is known as systemic risk. Since VaR and AVaR assess institutions' risk individually, both measures fail to consider an institution as part of a system, and therefore they do not consider spillover effects among institutions. To address this flaw, Adrian and Brunnermeier (2011) introduce the Conditional Value at Risk (CoVaR) risk metric. CoVaR is the VaR of a financial system based on whether or not institutions are in trouble. Several statistical methods have been applied to estimate the CoVaR risk measure. Adrian and Brunnermeier (2011) employ a simple quantile regression (QR) method, Borri et al. (2013) focus on the interconnectedness of financial system entities and the joint distribution of losses, Girardi and Ergün (2013) propose a multivariate generalized ARCH model, Bernardi et al. (2013b) employ a class of multivariate hidden Markov models, and Bernardi et al. (2013a) compute CoVaR using a quantile Bayesian regression approach.

In this paper, we estimate CoVaR using the same framework as in Adrian and Brunnermeier (2011). However, we address the estimation problem for regression parameters using the composite quantile regression (CQR) method. The CQR was first proposed by Zou et al. (2008) and has become popular in several fields related to the quantile approach. As CQR method simultaneously considers multiple quantile levels of a random variable, in contrast to the QR estimator that only assesses one quantile level, the results enjoy more accuracy. Indeed, the goal of the CQR is to propose an estimation method for assessing the conditional mean of the response by gathering information from multiple quantile levels instead of estimating the conditional quantiles individually.

The rest of the paper is organized as follows: Section 2 includes the methodology of applying the composite quantile method to estimating CoVaR. Section 3 is devoted to introducing two popular backtesting criteria for analyzing the efficiency of a method in estimating CoVaR risk measure. In Section 4 we estimate CoVaR for real market data using both quantile and composite quantile methods and compare the estimation results using the introduced backtesting criteria. The conclusion is presented in Section 5.

## 2 CoVaR Systemic Risk

Suppose that $Y_j$ and $Y_k$ are the interest variables of institutions or assets *j* and *k* ($k \neq j$), respectively, where $Y_j$ and $Y_k$ can be expressed through covariates $\boldsymbol{X} = (X_1, \ldots, X_M)$ such that $M \geq 1$. Assume that the behavior of the variable $Y_k$ is related to both covariates $\boldsymbol{X}$ and variable $Y_j$. So, $y_j$ and $y_k$, as the observations of $Y_j$ and $Y_k$, could be written as follow:

$$y_j = \mathrm{x}^\top \boldsymbol{\theta}_j + \epsilon_j,$$

$$y_k = \mathrm{x}^\top \boldsymbol{\theta}_k + \beta y_j + \epsilon_k,$$

where $\boldsymbol{\theta}_j$, $\boldsymbol{\theta}_k$, and $\beta$ are the unknown parameters of the regression models, and $\epsilon_j$ and $\epsilon_k$ are the independent error terms. Based on the definition of VaR, which is indeed a fixed quantile level (e.g., Philippe 2001), $\mathrm{VaR}_j^{\boldsymbol{X},\tau}$, which is used as the notation for the Value at Risk of institution *j* with the confidence level $\tau$ conditional on the observed $\boldsymbol{X}$, satisfies the following equation:

$$P(Y_j \leq \mathrm{VaR}_j^{\boldsymbol{X},\tau} | \boldsymbol{X} = \boldsymbol{x}) = \tau.$$

Similarly, $\mathrm{CoVaR}_{k|j}^{\boldsymbol{X},\tau}$ satisfies the below equation:

$$P(Y_k \leq \mathrm{CoVaR}_{k|j}^{\boldsymbol{X},\tau} | \boldsymbol{X} = \boldsymbol{x}, Y_j = \mathrm{VaR}_j^{\boldsymbol{X},\tau}) = \tau.$$

### 2.1 Quantile Regression Method in Estimating CoVaR

Sometimes, some problem-specific data features, such as skewness, fat-tails, outliers, etc., may affect the dependence between the variable of interest and the covariates. In this type of data, the least-squares method may show a weak performance in estimating the parameters of the regression model. The quantile regression is introduced by Koenker and Bassett (1978) to address this problem as a completely distribution-free and robust method for estimating the conditional quantiles of the response variable. This approach is rapidly expanding in statistics (e.g., Koenker 2005), social science (e.g., Firpo et al. 2009), biomedical (e.g., Flemming et al. 2017), finance (e.g., Demir et al. 2022), etc.

Let a regression model $\mathrm{y} = \mathrm{x}^\top \boldsymbol{\theta} + \boldsymbol{\epsilon}$ where y is the response variable, matrix x is the set of explanatory variables, $\boldsymbol{\theta}$ is the unknown parameter of the model, and $\boldsymbol{\epsilon}$ is the independent error term.

Using the representative $q_\tau(.)$ as the $\tau$-th quantile level, the following regression model is achieved:

$$q_\tau(\mathrm{y}) = \mathrm{x}^\top \boldsymbol{\theta}_\tau, \tag{1}$$

where $\boldsymbol{\theta}_\tau$ is the unknown parameter of the regression model related to the $\tau$-th quantile level. To estimate the unknown parameter of Eq. 1 (which is $\boldsymbol{\theta}_\tau$) using the QR method, the following optimization problem must be solved:

$$\hat{\boldsymbol{\theta}}_{QR} = \underset{\boldsymbol{\theta}QR}{\operatorname{argmin}} \sum_{i=1}^{T} \rho_\tau(y_i - \mathbf{x}_i^\top \boldsymbol{\theta}_{QR}), \tag{2}$$

where $(y_i, \mathrm{x}_i)$ for $i = 1, 2, \ldots, T$ are the observations of $(\mathrm{y}, \mathrm{x})$, and $\rho_\tau(u) = u(\tau - I(u \leq 0))$ is the check function defined by Koenker and Bassett (1978), in which $I$ is the indicator function. As the check function is not differentiable at zero, it is impossible to derive an explicit solution to the minimization problem 2; therefore, the estimation of $\theta_{QR}$ can be obtained by linear programming methods such as Newton or Alternating Direction Method of Multipliers (ADMM).

### 2.2 Time-Varying Quantile Regression Model of VaR and CoVaR

The quantile regression method has been widely considered as an approach for estimating VaR, AVaR, and CoVaR risk measures. In Huang (2013), a nonparametric quantile regression and a kernel estimator are used to find VaR estimates. In Chernozhukov and Du (2006), a conditional extremal quantile model is used to derive VaR estimates. Gerlach et al. (2011) uses dynamic conditional autoregressive quantile models to estimate VaR; Chen et al. (2017) use a quantile-function-valued time series approach to estimate VaR; Chun et al. (2012) apply least squares and quantile regression methods to estimate VaR and AVaR; and Bernardi et al. (2013a) uses Bayesian quantile regression to estimate CoVaR.

With a similar assumption of the previous sections, suppose that $(\boldsymbol{y}, \boldsymbol{x}) = (\boldsymbol{y}_t, \boldsymbol{x}_t)_{t=1}^T = (y_{j,t}, y_{k,t}, \boldsymbol{x}_t)_{t=1}^T$ is $T$ independent realizations of $(Y_j, Y_k, \boldsymbol{X})$. based on Adrian and Brunnermeier (2011), one-day-ahead $y_{j,t}$ and $y_{k,t}$ could be formulate as:

$$y_{j,t} = \mathrm{x}_{t-1}^\top \boldsymbol{\theta}_j + \epsilon_{j,t}, \tag{3}$$

$$y_{k,t} = \mathrm{x}_{t-1}^\top \boldsymbol{\theta}_k + \beta_t y_{j,t} + \epsilon_{k,t}. \tag{4}$$

By applying the quantile method to Eqs. 3 and 4, it is possible to write:

$$\mathrm{VaR}_{j,t}^{\boldsymbol{X},\tau} = \mathrm{x}_{t-1}^{\top}\boldsymbol{\theta}_j, \tag{5}$$

$$\mathrm{CoVaR}_{k|j,t}^{\boldsymbol{X},\tau} = \mathrm{x}_{t-1}^{\top}\boldsymbol{\theta}_k + \beta_t \mathrm{VaR}_{j,t}^{\boldsymbol{X},\tau}, \tag{6}$$

where $\boldsymbol{\theta}_j$, $\boldsymbol{\theta}_j$, and $\beta_t$ are unknown parameters of the model.

### 2.3 Composite Quantile Regression Method in Estimating CoVaR

According to Zou et al. (2008), the unknown error distribution in a regression model implies that relying on a single quantile level may result in inefficient parameter estimation. To achieve both robustness and high efficiency, they build upon the concept introduced by Koenker and Bassett (1978) by proposing the composite quantile regression method.

With a similar set up of the simple quantile model, let $\mathrm{y} = \mathrm{x}^{\top}\boldsymbol{\theta} + \boldsymbol{\epsilon}$ where $\mathrm{y}, \mathrm{x}, \boldsymbol{\theta}$, and $\boldsymbol{\epsilon}$ are the response variable, explanatory variables, unknown parameter of the model, and the independent error term, respectively. Denoting $0 < \tau_1 < \tau_2 < \cdots < \tau_K$ as a set of quantile levels, the unknown parameters of a regression model using the composite quantile method can be obtained by solving the below optimization problem:

$$(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_K, \hat{\boldsymbol{\theta}}_{CQR}) = \underset{b1, b_2, \ldots, b_K, \boldsymbol{\theta}_{CQR}}{\mathrm{argmin}} \sum_{k=1}^{K} [\sum_{i=1}^{T} \rho_{\tau_k}(y_i - b_k - \mathbf{x}_i^{'}\boldsymbol{\theta}_{CQR})], \tag{7}$$

where $\rho$ is the check function and $b_k$ is the $k-$th quantile of the error term. Typically, it is possible to use equally spaced quantile levels $\tau_k = \frac{k}{K+1}$ for $k + 1, 2, \ldots, K$. For example, one can set $K = 9$ and $K = 19$ to produce $0.1, 0.2, \ldots, 0.8, 0.9$ and $0.05, 0.1, 0.15, \ldots, 0.9, 0.95$ quantile levels.

Similar to Eqs. 2 and 7 is not differentiable at 0, so one cannot derive an explicit solution to this minimization problem. However, in Eq. 7, the minimization of the objective function is a convex optimization problem. It is shown that the composite quantile is uniquely defined and achieved by convex optimization techniques (e.g., Zou et al. 2008). Pietrosanu et al. (2017), for example, solve optimisationn problem 7 using three alternating direction method of multipliers (ADMM), majorize-minimization (MM), and coordinate descent (CD) algorithms.

### 2.4 Time-Varying Composite Quantile Regression Models of VaR and CoVaR

After introducing the CoVaR in 2011, quantile-based regression models have gained popularity in estimating this risk measure. However, this approach does not necessarily produce relevant results. Chun et al. (2012) question the performance of the simple quantile method, compared with the least-squares method, in estimating the quantile-based risk measures. Using the Monte Carlo simulation and generating four classes of distributions, it is shown that regardless of the error distribution, the least-squares method works better than the quantile method in estimating VaR and the AVaR risk measures. On the other hand, Zou et al. (2008) show that the relative efficiency of the composite quantile is greater than $70\%$ compared to the least squares, regardless of the error distribution. In this regard, we aim to apply the composite quantile method in estimating VaR and CoVaR to obtain more robust results than the simple quantile method. We review some concepts that deal with the VaR and its regression model that help us to define CoVaR in the composite quantile framework.

Consider a probability space $(\Omega, \mathcal{F}, P)$, and the space $\mathcal{Y} := L_p(\Omega, \mathcal{F}, P)$, with $p \in [1, \infty)$, of measurable random variables $Y : \Omega \longrightarrow \mathbb{R}$ having finite $p-th$ order moment. Artzner (1999) suggests that a risk risk measure $\rho : \mathcal{Y} \longrightarrow \mathbb{R}$ should satisfy four properties known as coherence axioms:

- (A1) Monotonicity: If $Y, Y^{'} \in \mathcal{Y}$ and $Y \succeq Y^{'}$ then $\rho(Y) \geqslant \rho(Y^{'})$,( The notation $Y \succeq Y^{'}$ means that $Y(\omega) \geqslant Y^{'}(\omega)$, for a.e. $\omega \in \Omega$).
- (A2) Convexity: $\rho(tY + (1-t)Y^{'}) \leq t\rho(Y) + (1-t)\rho(Y^{'})$ for all $Y, Y^{'} \in \mathcal{Y}$ and all $t \in [0, 1]$,
- (A3) Translation Equivariance: If $a \in \mathbb{R}$ and $Y \in \mathcal{Y}$, then $\rho(Y + a) = \rho(Y) + a$,
- (A4) Positive Homogeneity: If $t \geqslant 0$ and $Y \in \mathcal{Y}$, then $\rho(tY) = t\rho(Y)$.

Now, suppose that $\rho(.)$ is a law-invariant risk measure, which means $\rho(Y)$ depends only on the distribution of $Y$ (e.g., Kusuoka 2001), satisfying the axiom (A3). By applying $\rho(.)$ to Eq. 3, the following equation is obtained (see Chun et al. 2012 to proof):

$$\begin{aligned} \rho_{|\boldsymbol{X}}(y_{j,t}) &= \rho_{|\boldsymbol{X}}(\boldsymbol{x}_{t-1}^{\top}\boldsymbol{\theta}_j + \epsilon_{j,t}) \\ &= \boldsymbol{x}_{t-1}^{\top}\boldsymbol{\theta}_j + \rho_{|\boldsymbol{X}}(\epsilon_{j,t}) \\ &= \boldsymbol{x}_{t-1}^{\top}\boldsymbol{\theta}_j + \rho(\epsilon_{j,t}). \end{aligned}$$

Both VaR and CoVaR are law-invariant risk measure (e.g., Föllmer and Knispel 2013), so by applying VaR to Eq. 3 and CoVaR to Eq. 4, we obtain:

$$\mathrm{VaR}_{j,t}^{\boldsymbol{X},\tau} = \mathrm{x}_{t-1}^{\top}\boldsymbol{\theta}_j + \mathrm{VaR}^{\boldsymbol{X},\tau}(\epsilon_{j,t}), \tag{8}$$

$$\mathrm{CoVaR}_{k|j,t}^{\boldsymbol{X},\tau} = \mathrm{x}_{t-1}^{\top}\boldsymbol{\theta}_k + \beta_t \mathrm{VaR}_{j,t}^{\boldsymbol{X},\tau} + \mathrm{VaR}^{\boldsymbol{X},\tau}(\epsilon_{k,t}), \tag{9}$$

where $\boldsymbol{\theta}_j$, $\boldsymbol{\theta}_k$, and $\beta$ are the unknown parameters of the model and $\mathrm{VaR}^{\boldsymbol{X},\tau}(\epsilon)$ is the $\tau$-th quantile level of the distribution of $\epsilon$. The superiority of the composite quantile method is that this method estimates the different quantiles of the $\epsilon$, including $\tau - th$ quantile. Thanks to this advantage, one can rewrite Eqs. 8 and 9 in the composite quantile framework as:

$$\mathrm{VaR}_{j,t}^{\boldsymbol{X},\tau} = \mathrm{x}_t^{\top}\boldsymbol{\theta}_j + b_{j_\tau}(t),$$

$$\mathrm{CoVaR}_{k|j,t}^{\boldsymbol{X},\tau} = \mathrm{x}_t^{\top}\boldsymbol{\theta}_k + \beta_t \mathrm{VaR}_{j,t}^{\boldsymbol{X},\tau} + b_{k_\tau}(t),$$

where $\boldsymbol{\theta}_j$, $\boldsymbol{\theta}_k$, and $\beta_t$ are the unknown parameters of the regression model and $b_{j_\tau}$ and $b_{k_\tau}$ are the $\tau$-th quantiles of $\epsilon_j$ and $\epsilon_k$ computed using the optimization problem 7.

## 3 Backtesting of CoVaR Risk Measure

Along with the increased application of quantile-based risk measures such as VaR, AVaR, and CoVaR in the financial markets, economists and statisticians have started introducing backtesting criteria to evaluate the effectiveness of various methods for estimating these risk measures. The amount of backtesting for the quantile-based risk measures is considerable, and each criterion pursues a particular goal. For instance, Angelidis and Degiannakis (2018) propose a two-stage backtesting method that leverages conditional volatility models, while Christoffersen and Pelletier (2004) utilize the time between VaR violations as a backtesting approach. We focus on two widely used backtesting criteria, violation and risk map, to achieve a rational comparison between quantile and composite quantile methods in estimating CoVaR.

### 3.1 Violation Backtesting

Violation is one of the most popular benchmarks, with a very easy-to-apply procedure accepted by many researchers to evaluate the efficiency of a method in estimating VaR and CoVaR. Violation is said to occur whenever an asset return on a particular day exceeds the estimated CoVaR on the same day (e.g., McNeil and Frey 2000). The violation process is defined as:

$$I_\tau(t) = \begin{cases} 1 & \text{if} \quad r_t < \text{CoVaR}_t^\tau, \\ 0 & \text{else}, \end{cases} \tag{10}$$

where $r_t$ is the return at time $t$. Christoffersen (1998) studies the Unconditional Coverage (UC) hypothesis in detail. Based on the UC hypothesis, the probability of the exceeded returns must be equal to the $\tau$ coverage rate:

$$\Pr[I_\tau(t) = 1] = \mathbb{E}[I_\tau(t)] = \tau.$$

In this regard, if the violation of method A be closer to $\tau$ compared to method B, it is said that method A is more efficient than method B in estimating VaR risk measure. Two below scenarios happen to analyze the performance of a method:

$$\begin{cases} \text{if } \Pr[I_\tau(t) = 1] > \tau; & \text{the risk is} \quad \text{underestimated}, \\ \text{if } \Pr[I_\tau(t) = 1] < \tau; & \text{the risk is} \quad \text{overestimated}. \end{cases} \tag{11}$$

Since CoVaR is a special VaR, a similar approach is implemented for CoVaR. Hence, for comparing the efficiency of the quantile and composite quantile methods for a fixed level of confidence level, it is only enough to compare their violations for that confidence level.

### 3.2 Risk Map

In contrast to its popularity in analyzing the performance of the quantile-based risk measures, the violation criterion suffers from an undeniable drawback. This criterion only focuses on the number of exceeded returns and does not consider the magnitude of these losses. However, in practice, market participants are also paying attention to the magnitude of their losses. Berkowitz and O'Brien (2002) and Stulz (2008) discuss this issue more profoundly.

To address the magnitude issue, Colletaz et al. (2013) introduce the risk map, a novel criterion of violations that accounts for the number and magnitude of extreme losses. In this framework, they use the benefits of an extra $\tau^{'}$ confidence level which is much lower than the main confidence level $\tau$( e.g. $\tau = 1\%$ and $\tau^{'} = 0.2\%$). If a significant loss happens, this loss not only exceeds the estimated CoVaR for $\tau$ but also is likely to exceed the estimated CoVaR for $\tau^{'}$ confidence level. The first one is named the exceptions and the latter one is the super exceptions.

Let $\{I_t(\tau)\}_{t=1}^{T}$ and $\{I_t(\tau^{'})\}_{t=1}^{T}$ be the violation process for $\tau$ and $\tau^{'}$ confidence levels defined in Eq. 10. Based on the Unconditional Coverage hypothesis, the below equations must be satisfied:

$$\Pr[I_\tau(t) = 1] = \mathbb{E}[I_\tau(t)] = \tau,$$

and

$$\Pr[I_{\tau^{'}}(t) = 1] = \mathbb{E}[I_{\tau^{'}}(t)] = \tau^{'}.$$

Let $N$ and $N^{'}$ be the number of the CoVaR violations for $\tau$ and $\tau^{'}$, respectively:

$$N = \sum_{t=1}^{T} I_\tau(t) \qquad N^{'} = \sum_{t=1}^{T} I_{\tau^{'}}(t).$$

Using the null and alternative hypothesis, one can test the UC hypothesis for the CoVaR violations as:

$$H_0 : \mathbb{E}[I_\tau(t)] = \tau,$$
$$H_1 : \mathbb{E}[I_\tau(t)] \neq \tau.$$

It is shown that $LR_{UC}$ is an asymptotically chi-square distribution with one degree of freedom (e.g., Wipplinger 2007). Under the null hypothesis, the corresponding log-likelihood ratio statistics is defined as:

$$LR_{UC}(\tau) = -2\ln\left[(1-\tau)^{T-N}\tau^{N}\right] + 2\ln\left[(1-\frac{N}{T})^{T-N}(\frac{N}{T})^{N}\right] \xrightarrow[T\longrightarrow\infty]{d} \chi^2(1). \quad (12)$$

A similar validation test can be conducted for the super exceptions:

$$H_0 : \mathbb{E}[I_{\tau}^{'}(t)] = \tau^{'},$$
$$H_1 : \mathbb{E}[I_{\tau}^{'}(t)] \neq \tau^{'}.$$

Similar to Eq. 12, one can define $LR_{UC}(\tau^{'})$ by replacing $\tau$ and $N$ by $\tau^{'}$ and $N,^{'}$, respectively.

The goal of the risk map is to merge the non-rejection zone of $LR_{UC}(\tau)$ and $LR_{UC}(\tau^{'})$ for a fixed sample size. Colletaz et al. (2013) create a balance between the exceptions and the super exceptions by offering a graphical way. In this regard, the null hypothesis is defined to jointly test the number of CoVaR exceptions and super exceptions:

$$H_0 : \mathbb{E}[I_{\tau}(t)] = \tau \quad \text{and} \quad \mathbb{E}[I_{\tau}^{'}(t)] = \tau^{'}.$$

The Multivariate Unconditional Coverage (MUC) test is defined as:

$$LR_{MUC}(\tau, \tau^{'}) = -2\ln\left[(1-\tau)^{N_0}(\tau - \tau^{'})^{N_1}(\tau^{'})^{N_2}\right] + 2\ln\left[(\frac{N_0}{T})^{N_0}(\frac{N_1}{T})^{N_1}(\frac{N_2}{T})^{N_2}\right] \xrightarrow[T\longrightarrow\infty]{d} \chi^2(2),$$

where $N_i = \sum_{t=1}^{T} J_{i,t}$ for $i = 0, 1, 2$. Indicators $J_{i,t}$ for $i = 0, 1, 2$ are calculated using below formulas:

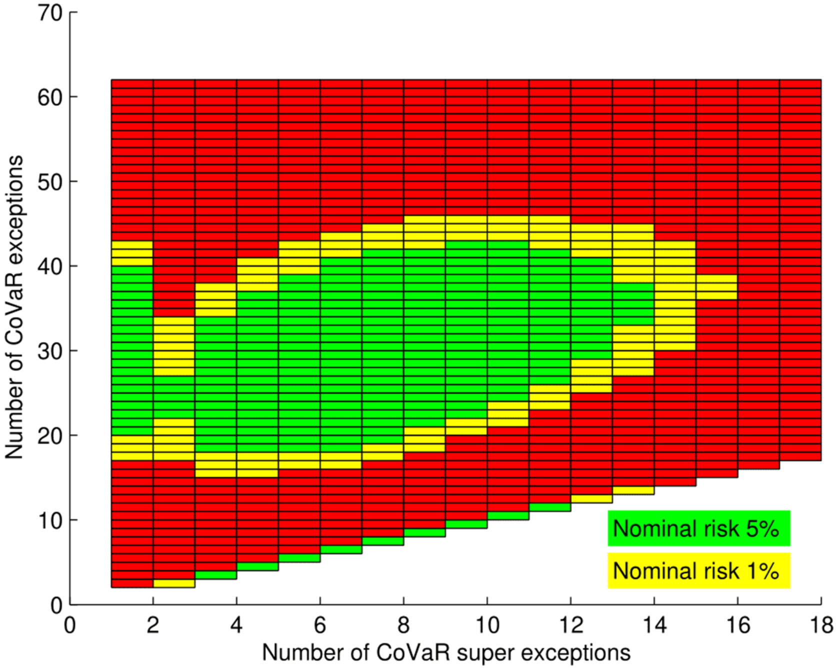
$$J_{1,t} = I_{\tau}(t) - I_{\tau'}(t) = \begin{cases} 1 & \text{if} \quad \mathrm{CoVaR}_t^{\tau^{'}} < r_t < \mathrm{CoVaR}_t^{\tau}, \\ 0 & \text{otherwise.} \end{cases}$$

$$J_{2,t} = I_{\tau'}(t) = \begin{cases} 1 & \text{if} \quad r_t < \mathrm{CoVaR}_t^{\tau^{'}}, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$J_{0,t} = 1 - J_{1,t} - J_{2,t} = 1 - I_{\tau}(t).$$

A risk map is conducted based on the acceptance and rejection of two confidence levels. If the pair $(N, N^{'})$ falls into the green area, one cannot reject the multivariate null hypothesis $\mathbb{E}[I_{\tau}(t)] = \tau$ and $\mathbb{E}[I_{\tau}^{'}(t)] = \tau^{'}$ under the 95% confidence level. If $(N, N^{'})$ falls into a yellow cell, one can reject the null hypothesis at the 95% confidence level but cannot reject at the 99%. Finally, a red cell implies that one can reject the null hypothesis at both 95% and 99% confidence levels (Fig. 1).



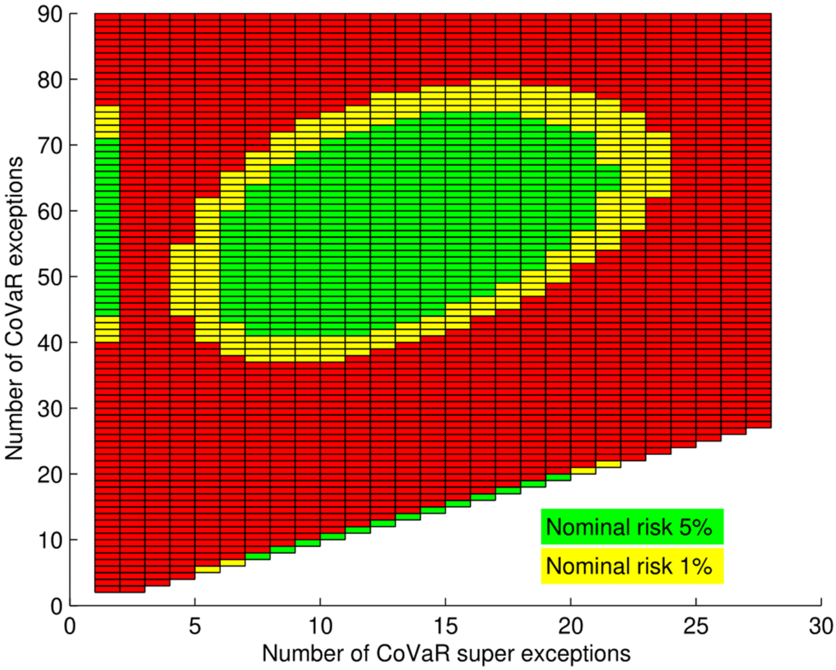




**Fig. 1** The risk maps for different numbers of CoVaR exceptions (N) and CoVaR super exceptions $(N^{'})$ for $T = 555$

## 4 Empirical Studies

This section applies the methodologies discussed in the previous sections to real market data. In line with Bernardi et al. (2013a), the data are obtained from the different sectors of the Standard and Poor 500 Composite Index ($S\&P$ 500) on the basis of daily observations. The collected data include the 2018 financial crisis to consider a crisis period. Many stock markets around the world saw their values fall in 2018. For example, the $S\&P$ 500 index finished the year down 6 percent, the Dow Jones Industrial Average dropped 5.6 percent, and the NASDAQ composite slid nearly 4 percent. Moreover, most European and Asian markets also lost ground in 2018.

Our empirical analysis is based on publicly traded U.S. companies of $S\&P$ 500 listed in Table 1. The companies belong to different sectors including Financial, Consumer, Energy, Industrial, Technology. Daily equity price data included a sample period from January 3, 2017 to December 31, 2024 (Fig. 2).

We use the below indexes as the explanatory variables of the regression models (Table 2):

REIT Index is securitized portfolios of real estate properties which allows ordinary investors to buy shares in commercial real estate portfolios, the MSCI EAFE Index captures several large and mid-cap representation in 21 countries, and GCG Index includes the futures contracts for a diversified group of commodities futures (Fig. 3).

### 4.1 CoVaR Estimating and Backtesting Using Quantile and Composite Quantile Methods

We perform out-of-sample backtesting procedures to compare the efficiency of the quantile and composite quantile methods in estimating CoVaR. We consider observations from January 3, 2017, to December 31, 2024, comprising 2012 daily returns. The full data period is divided into a learning sample: January 3, 2017, to May 5, 2017; and a forecasting sample: May 6, 2017, to December 31, 2024. We employ an

**Table 1** Summary statistics of the company's stock prices

| Name | Ticker | Sector | Mean | Min | Max | Std. Dev |
|---|---|---|---|---|---|---|
| American Express Co. | AXP | Financial | 141.378 | 68.960 | 305.570 | 50.706 |
| Bank of America Corp. | BAC | Financial | 32.146 | 18.080 | 49.380 | 6.597 |
| Comerica Inc. | CMA | Financial | 67.644 | 26.050 | 102.210 | 17.564 |
| JPMorgan Chase & Co. | JPM | Financial | 131.967 | 79.030 | 250.290 | 36.003 |
| KeyCorp | KEY | Financial | 17.332 | 8.160 | 27.010 | 3.621 |
| Moody's Corp. | MCO | Financial | 269.412 | 94.270 | 500.880 | 100.080 |
| Morgan Stanley | MS | Financial | 69.372 | 27.810 | 134.990 | 23.938 |
| McDonald's Corp. | MCD | Consumer | 220.385 | 119.480 | 316.560 | 49.059 |
| Nike Inc. | NKE | Consumer | 98.659 | 50.830 | 177.510 | 30.057 |
| Chevron Corp. | CVX | Energy | 126.925 | 54.220 | 188.050 | 27.107 |
| Exxon Mobil Corp. | XOM | Energy | 81.419 | 31.450 | 125.370 | 22.964 |
| General Electric Co. | GE | Industrial | 78.601 | 27.333 | 194.230 | 40.899 |
| Intel Corp. | INTC | Technology | 44.087 | 18.890 | 68.470 | 11.057 |
| Standard and Poor 500 | $S\&P$ 500 | Index | 3675.050 | 2237.400 | 6090.270 | 977.332 |

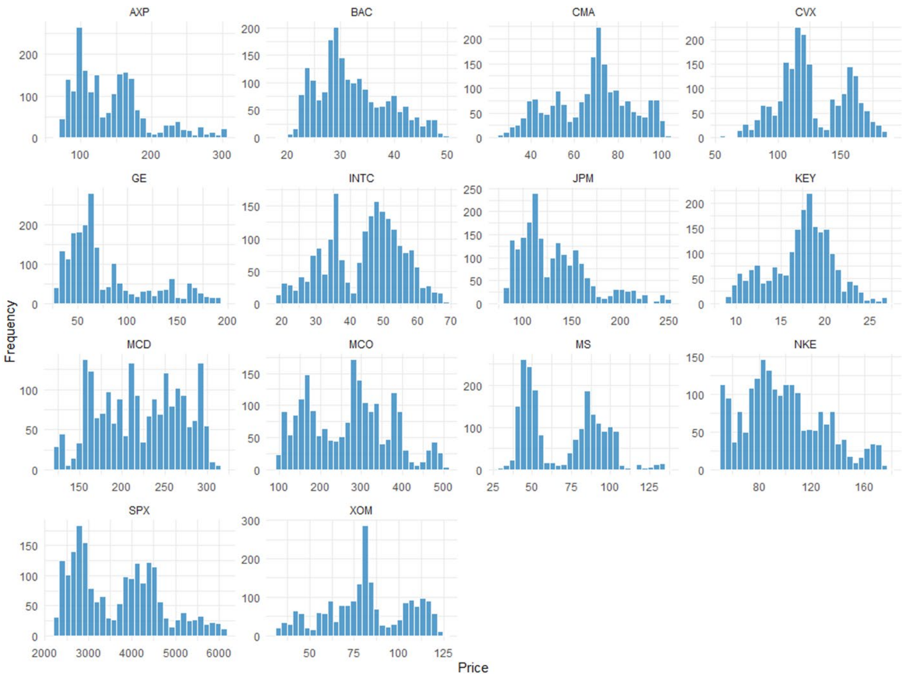



**Fig. 2** Histogram of daily prices for the listed U.S. stocks

**Table 2** Summary statistics of the indexes

| Name | Ticker | Mean | Min | Max | Std. Dev |
|---|---|---|---|---|---|
| Real Estate Investment Trust | REIT | 2162.484 | 1380.150 | 2964.060 | 352.897 |
| MSCI EAFE | MSCI EAFE | 2035.158 | 1354.300 | 2506.690 | 205.939 |
| iShares S&P GSCI Commodity | GCG | 17.274 | 7.830 | 26.300 | 3.742 |

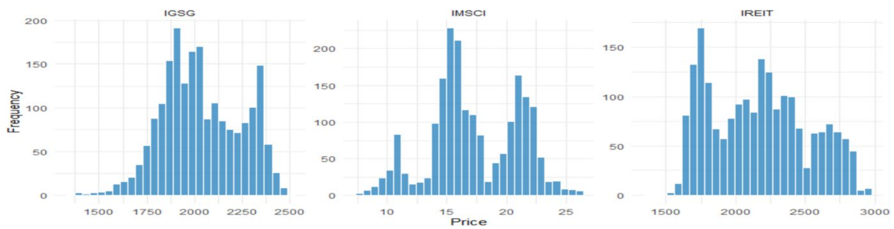



**Fig. 3** Histogram of the indexes

estimation window of 100 observations to estimate one-day-ahead VaR, and an estimation window with the same length to estimate one-day-ahead CoVaR using both quantile and composite quantile methods. This procedure is consistent with both vio-

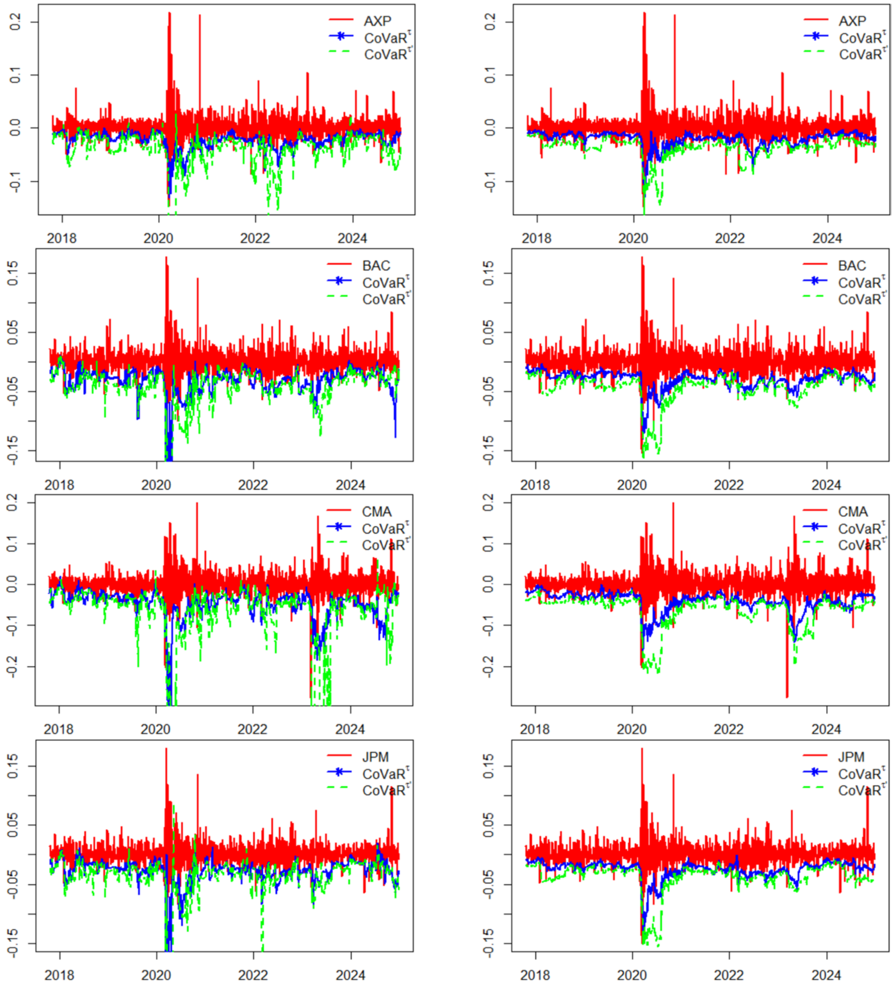



**Fig. 4** Time series plot of the CoVaR for companies at the confidence levels $\tau = 0.05$( blue line), and $\tau' = 0.01$( green line), using the quantile method

lation and risk map backtesting. The only difference is that for the violation, one must slide the window for one confidence level, while in the risk map, the window sliding procedure must be performed for two confidence levels to compute both exceptions and super exceptions. The below figures are the estimated CoVaR for several U.S companies with two quantile and composite quantile methods to obtain violation and risk map criteria (Figs. 4 and 5).

### 4.1.1 Violation

Table 3 shows violations of the CoVaR at the different confidence levels for companies listed in Table 1.

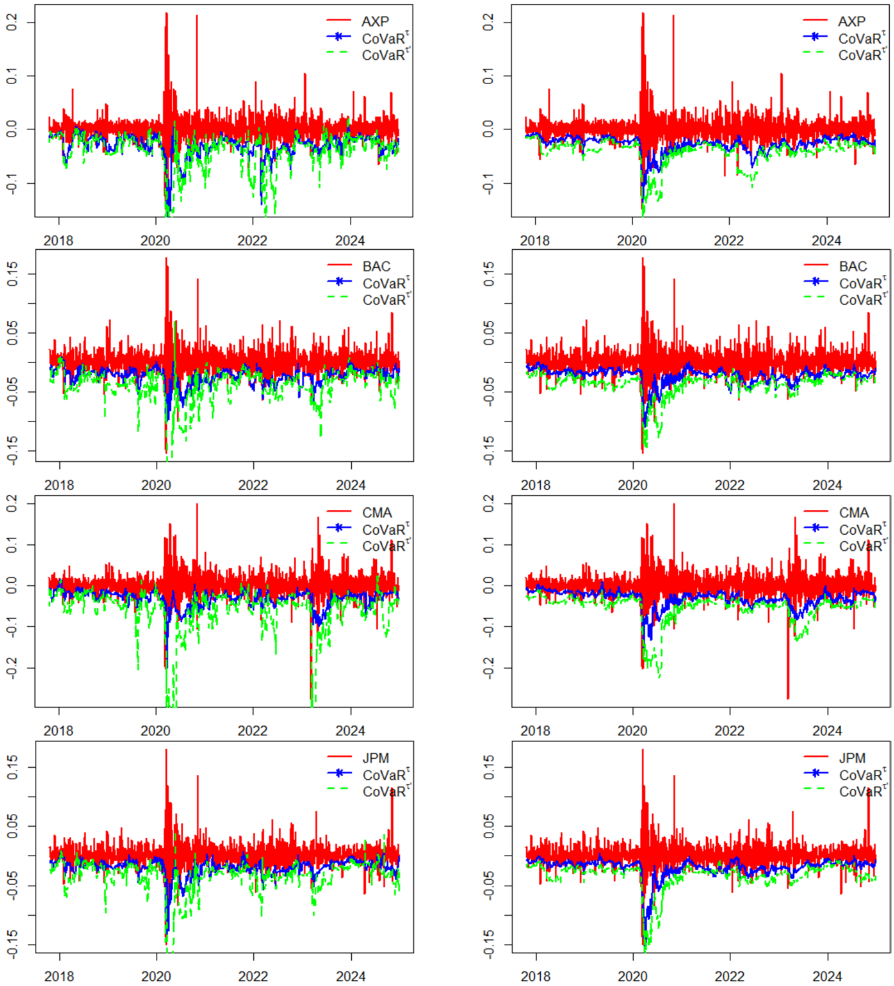



**Fig. 5** Time series plot of the CoVaR for companies at the confidence levels $\tau = 0.02$( blue line), and $\tau' = 0.1$( green line), using the quantile method

Comparing the results in Table 3 reveal that the performance of the composite quantile approach is far better than the quantile method for all companies at any supposed confidence level. Especially in the lower quantile levels, this efficiency is more compared to upper levels. Note that both risk measures underestimate CoVaR in almost all cases. Until this point, using the violation backtesting, we show that the composite quantile method enjoys considerable efficiency compared to the simple quantile method. We analyze both methods using the risk map criterion in the next steps.

**Table 3** Violations of CoVaR at confidence levels $\tau = 0.01, 0.02, 0.05$, and $0.1$

| Ticker | Method \ Confidence Level | 0.01 | 0.02 | 0.05 | 0.1 |
|---|---|---|---|---|---|
| AXP | Q | 0.0534 | 0.0660 | 0.1113 | 0.1475 |
| | CQ | 0.0175 | 0.0260 | 0.0806 | 0.1240 |
| BAC | Q | 0.0605 | 0.0726 | 0.0767 | 0.1224 |
| | CQ | 0.0193 | 0.0285 | 0.0809 | 0.1294 |
| CMA | Q | 0.0550 | 0.0518 | 0.0804 | 0.1454 |
| | CQ | 0.0190 | 0.0421 | 0.0863 | 0.1309 |
| JPM | Q | 0.0641 | 0.0696 | 0.0931 | 0.1349 |
| | CQ | 0.0194 | 0.0319 | 0.0826 | 0.1242 |
| KEY | Q | 0.0499 | 0.0533 | 0.0822 | 0.1314 |
| | CQ | 0.0211 | 0.0304 | 0.0734 | 0.1241 |
| MCO | Q | 0.0352 | 0.0768 | 0.1023 | 0.1263 |
| | CQ | 0.0201 | 0.0311 | 0.0812 | 0.1236 |
| MS | Q | 0.0518 | 0.0489 | 0.0741 | 0.1527 |
| | CQ | 0.0210 | 0.0337 | 0.0451 | 0.1248 |
| MCD | Q | 0.0679 | 0.0662 | 0.0915 | 0.1290 |
| | CQ | 0.0175 | 0.0301 | 0.0690 | 0.1260 |
| NKE | Q | 0.0735 | 0.0490 | 0.0470 | 0.1492 |
| | CQ | 0.0230 | 0.0320 | 0.0735 | 0.1205 |
| CVX | Q | 0.0662 | 0.0627 | 0.1260 | 0.1756 |
| | CQ | 0.0275 | 0.0485 | 0.0662 | 0.1132 |
| XOM | Q | 0.0771 | 0.0753 | 0.0915 | 0.1600 |
| | CQ | 0.0140 | 0.0303 | 0.0753 | 0.1312 |
| GE | Q | 0.0573 | 0.0537 | 0.0700 | 0.1492 |
| | CQ | 0.0230 | 0.0303 | 0.0771 | 0.1168 |
| INTC | Q | 0.0609 | 0.0627 | 0.0987 | 0.1816 |
| | CQ | 0.0175 | 0.0340 | 0.0717 | 0.1366 |

#### 4.1.2 Risk Map

Table 4 shows the numbers of CoVaR exceptions ($N$) at confidence levels 0.05, 0.1, and Numbers of CoVaR super exceptions ($N^{'}$) at the confidence levels 0.01, 0.02.

Based on Table 4, we conclude that the composite quantile method estimates the CoVaR risk measure much better than the quantile approach for all listed companies. In the composite quantile method, we cannot reject the multivariate null hypothesis $\tau = 0.05$ and $\tau^{'} = 0.01$ at the 95% confidence level for AXP, BAC, KEY, MCO, MS, MCD, NKE, CVX, XOM, GE, and INTC companies. Also, we cannot reject for CMA, MCO, and NKE at the 99% confidence level. For $\tau = 0.1$ and $\tau^{'} = 0.02$, we observe an interesting pattern. In the composite quantile method, we cannot reject the null hypothesis at the 99% confidence level for all companies except AXP and INTC, where we can reject it at both the 99% level but not at 95% level. Interestingly, with the quantile method, we can reject the multivariate null hypothesis $\tau = 0.05$ and $\tau^{'} = 0.01$ at both 95% and 99% confidence levels for all companies. Similarly, we can reject the null hypothesis for $\tau = 0.1$ and $\tau^{'} = 0.02$ at both the 95% and 99% confidence levels for all companies.

**Table 4** Numbers of CoVaR exceptions ($N$) at confidence levels 0.05, 0.1, and Numbers of CoVaR super exceptions ($N^{'}$) at confidence levels 0.01, 0.02.

| Ticker | Method \ Confidence Level | $\tau = 0.05$ $\tau^{'} = 0.01$ | $\tau = 0.1$ $\tau^{'} = 0.02$ |
|---|---|---|---|
| AXP | Q | | |
| | CQ | | |
| BAC | Q | | |
| | CQ | | |
| CMA | Q | | |
| | CQ | | |
| JPM | Q | | |
| | CQ | | |
| KEY | Q | | |
| | CQ | | |
| MCO | Q | | |
| | CQ | | |
| MS | Q | | |
| | CQ | | |
| MCD | Q | | |
| | CQ | | |
| NKE | Q | | |
| | CQ | | |
| CVX | Q | | |
| | CQ | | |
| XOM | Q | | |
| | CQ | | |
| GE | Q | | |
| | CQ | | |
| INTC | Q | | |
| | CQ | | |

## 5 Conclusion

This paper contributed to the literature by addressing the problem of estimating the CoVaR risk measure using a robust composite quantile regression method. In the most of the financial papers, CoVaR risk measure is estimated and analyzed by using the simple quantile method. By emphasizing the drawback of the simple quantile method in estimating quantile-based risk measures, a new method, called composite quantile method, was suggested to assess CoVaR systemic risk. Using two backtesting criteria, violation and risk map, the out-of-sample studies on companies from different sectors of the U.S stock market, including financial, consumer goods, energy, industries, and technology, revealed that the composite quantile method is superior to the simple quantile method in estimating CoVaR. Based on the results, we recommend market participants and companies consider the composite quantile method as a robust approach to estimating CoVaR systemic risk.

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## Declarations

**Conflicts of Interest** The authors have no relevant financial or non-financial interests to disclose.



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