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Article

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Norm estimates for the Hilbert matrix operator on weighted Bergman spaces

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A B S T R A C T

We study the Hilbert matrix operator H and a related integral operator T acting on the standard weighted Bergman spaces A^p_α. We obtain an upper bound for T, which yields the smallest currently known explicit upper bound for the norm of H for $-1 < \alpha < 0$ and $2+\alpha < p < 2(2+\alpha)$. We also calculate the essential norm for all $p > 2+\alpha > 1$, extending a part of the main result in [Adv. Math. 408 (2022) 108598] to the standard unbounded weights. It is worth mentioning that except for an application of Minkowski's inequality, the norm estimates obtained for T are sharp.

1. Introduction

The Hilbert matrix operator on Banach spaces of analytic functions is defined, using the Taylor series, as

$$\sum_{n\geq 0} a_n z^n \mapsto \sum_{n\geq 0}\left(\sum_{k\geq 0}\frac{a_k}{n+k+1}\right)z^n.$$

The norm of the Hilbert matrix operator H on Banach spaces of analytic functions on the unit disk is a well studied topic. On Hardy spaces H^p, the conjecture

$$\|H\|_{\mathcal{L}(H^p)} = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}, \quad 1 < p < \infty$$

was solved in the positive in [8]. The proof implies that the essential norm, $\|H\|_{\mathcal{L}(H^p)}$, is the same value as the norm. On classical Korenblum spaces H^∞_α [3, Theorem 3.3], it was proved that

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$$\|H\|_{\mathcal{L}(H_\alpha^\infty)} = \frac{\pi}{\sin(\alpha\pi)},\ 0 < \alpha \le \alpha_0$$

and that

$$\frac{\pi}{\sin(\alpha\pi)} < \|H\|_{\mathcal{L}(H_\alpha^\infty)} < [1 + (2\alpha - 1)(2^{1-\alpha} - 1)]\frac{\pi}{\sin(\alpha\pi)}, \quad \alpha_0 < \alpha < 1,$$

where $\alpha_0 \in]\frac{1}{2}, 1[$ is the unique zero to $\alpha \mapsto \frac{2\alpha-1}{2}\beta(\frac{1}{2}, 1-\alpha) - 1$, where β is the classical beta function. The explicit expression for the norm is [3, Theorem 3.1]

$$\|H\|_{\mathcal{L}(H_\alpha^\infty)} = \sup_{r\in]0,1[} (1+r)^\alpha \int_0^1 \frac{(1-rt)^{2\alpha-1}}{t^\alpha[2-(1+r)t)]^\alpha}\, dt.$$

Replacing the weight $(1-|z|^2)^\alpha$ by the equivalent weight $w_\alpha(z) = (1-|z|)^\alpha$, it was proved in [13] that

$$\|H\|_{\mathcal{L}(H_{w_\alpha}^\infty)} = \|H\|_{e,\mathcal{L}(H_{w_\alpha}^\infty)} = \|H\|_{e,\mathcal{L}(H_\alpha^\infty)} = \frac{\pi}{\sin(\alpha\pi)},\ 0 < \alpha < 1.$$

Concerning the norm of H on the standard weighted Bergman spaces A_α^p, the following conjecture was stated in [11]:

Conjecture 1. If $1 < 2 + \alpha < p$, then

$$\|H\|_{\mathcal{L}(A_\alpha^p)} = \frac{\pi}{\sin\left(\frac{(2+\alpha)\pi}{p}\right)}.$$

The conjecture is still believed to hold true, although it has not been fully resolved yet. The first result dates back to 2004 when it was proved in [5] that the previously known representation

$$H(f)(z) = \int_0^1 \frac{f(t)}{1-zt}\, dt, \quad z \in \mathbb{D}$$

for the operator acting on Hardy spaces [6] is still valid on Bergman spaces. This integral expression can already be seen in [16], which was published in 1950. In [5] the author examined $H\colon A_\alpha^p \to A_\alpha^p$ when $\alpha = 0$ (the unweighted case), and proved it to be bounded for $p > 2$ and that the value given in Conjecture 1 is an upper bound for $\|H\|_{\mathcal{L}(A_\alpha^p)}$ when $p \ge 4$. In [8] the correct lower bound for the norm was found and in [1] the conjecture was solved for the case $\alpha = 0$. Around the same time, the boundedness of $H\colon A_\alpha^p \to A_\alpha^p,\ p > 2 + \alpha > 1$ was proved in [10]. About a year later, [11] it was proved that the value given in Conjecture 1 is indeed a lower bound for $\|H\|_{\mathcal{L}(A_\alpha^p)}$ when $1 < 2 + \alpha < p$ extending earlier results on the lower bound [8] and solving one side of Conjecture 1. Concerning the upper bound, progress has been made in [5,8,10,11,15,12,7,4], but it is still open even in the Hilbert case when $p = 2$, in which case $-1 < \alpha < 0$.

When $\alpha > 0$, Conjecture 1 has been proved for $p > 2 + \alpha$ satisfying

$$p \ge \alpha + 2 + \sqrt{(\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)},$$

see [12, Corollary 1.1], or

$$p \ge \frac{3\alpha}{4} + 2 + \sqrt{\left(\frac{3\alpha}{4} + 2\right)^2 - \frac{\alpha+2}{2}},$$

see [7, Theorem 1.1] (see also [4, Theorem 3.2]). The constraint given in [7,4] solves the conjecture for new pairs (p, α) compared to [12] when $\alpha \geq \alpha_0 \approx \frac{1}{2}$. Moreover, in [4, Section 5 and Theorem 4.1], the conjecture was proved to hold whenever $\alpha = 1$ or $0 < \alpha < \frac{1}{47}$ and $p > 2 + \alpha$. Among other results in [4], the conjecture is also true when $2 + \alpha < p < \beta_\alpha,\ \alpha > 0$, for some $2 + \alpha < \beta_\alpha < \frac{5}{2} + \alpha$ [4, Theorem 3.8].

Additionally, for all $p > 2 + \alpha \geq 2$, the essential norm [13] is given by the conjectured value for the norm, that is,

$$\|H\|_{e,\mathcal{L}(A^p_\alpha)} = \frac{\pi}{\sin\left(\frac{(2+\alpha)\pi}{p}\right)}. \tag{1.1}$$

When $-1 < \alpha < 0$ and $p > 2 + \alpha$, various upper bounds were obtained in [12,2]. Very recently, the conjecture was solved in the positive [4, Theorem 6.5] for $p \geq 2(2 + \alpha),\ -1 < \alpha < 0$.

In this paper, we extend (1.1) to hold true for all $1 < 2 + \alpha < p$, see Theorem 1.2. For $\alpha < 0$ Conjecture 1 remains unsettled for $2 + \alpha < p < 2(2 + \alpha)$, including all the weighted Hilbert Bergman spaces on which H is bounded, but an improved upper bound, compared to [2] and [12], is obtained using a new approach, see Theorem 1.1. The theorem is also an improvement of the previously known bounds for e.g. $p = \alpha + 3$ when $\alpha > 0$ is large, see Remark 3.6.

Another interesting result is Theorem 3.8, which gives a lower bound for the norm of the extended Hilbert matrix operator T defined in (2.2). It is also worth noting that a new phenomenon occurs when examining T compared to the classical Hilbert matrix operator H due to the extra singularity. The limit of the norm of the operator acting on a weakly null sequence is not invariant under Minkowski's inequality, see Remark 3.7. Without this phenomenon, Conjecture 1 would likely have been solved by Theorem 1.1 due to the sharpness of Theorem 3.4. The two main results are presented below:

Theorem 1.1. *If $p > 2 + \alpha > 1$, then*

$$\|H\|_{\mathcal{L}(A^p_\alpha)} \leq \int_{-1}^{1} \frac{\left(\frac{1}{2}\left[\frac{1}{(1+t)^{4+2\alpha-p}} + \frac{1}{(1-t)^{4+2\alpha-p}}\right]\right)^{\frac{1}{p}}}{(1-t^2)^{1-\frac{2+\alpha}{p}}}\, dt < 2^{1-\frac{1}{p}} \frac{\pi}{\sin\left(\pi\frac{2+\alpha}{p}\right)}.$$

Theorem 1.2. *For $p > \alpha + 2 > 1$,*

$$\|H\|_{e,\mathcal{L}(A^p_\alpha)} = \frac{\pi}{\sin\left(\pi\frac{2+\alpha}{p}\right)}.$$

2. Preliminaries

For $p \geq 1$ and $\alpha > -1$, the weighted Bergman space A^p_α is the Banach space of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that $\|f\|_{A^p_\alpha} < \infty$, where

$$\|f\|_{A^p_\alpha} := \left(\int_{\mathbb{D}} |f|^p\, dA_\alpha\right)^{\frac{1}{p}}$$

and $dA_\alpha(x + iy) = (1 + \alpha)(1 - |x + iy|^2)^\alpha \frac{dx\,dy}{\pi}$. It is well known that the point evaluations are bounded on A^p_α, more precisely,

$$|f(z)| \lesssim (1 - |z|^2)^{-\frac{2+\alpha}{p}},\ z \in \mathbb{D}. \tag{2.1}$$

We define the Hilbert matrix operator and the extended Hilbert matrix operator to be

$$H(f)(z) = \int_0^1 \frac{f(x)}{1-xz}\,dx \quad \text{and} \quad T(f)(z) = \int_{-1}^1 \frac{f(t)}{1-tz}\,dt, \quad z \in \mathbb{D}, \tag{2.2}$$

respectively. Furthermore, we define the following automorphisms of $\mathbb{D}$:

$$S_a(z) := \frac{z-a}{1-z\overline{a}} \quad \text{yielding} \quad S_a^{-1}(z) = \frac{z+a}{1+z\overline{a}} \text{ and } 1 - |S_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}, \quad a, z \in \mathbb{D}.$$

We also define $f_{c,\theta}$, $0 < c < \frac{2+\alpha}{p}, \theta \in [0,1]$ to be the normalized version of the following convex combination of approximate evaluation functions on A^p_α

$$\hat{f}_{c,\theta} \colon z \mapsto \theta(1+z)^{-c} + (1-\theta)(1-z)^{-c}.$$

For a given $z \in \mathbb{D}$, we can apply the change of variables $t \mapsto S_t(z)$ to obtain

$$T(f)(z) = \int_{-1}^1 \frac{f(S_t(z))}{1-zt}\,dt. \tag{2.3}$$

This can be justified by first considering real z, integration on $]-r, r[, 0 < r < 1$ and using (2.1) to justify the limit $r \to 1$. Then expansion to arbitrary $z \in \mathbb{D}$ by analytic expansion.

From

$$T(f)(z) = \int_0^1 \frac{f(-t)}{1+tz}\,dt + \int_0^1 \frac{f(t)}{1-tz}\,dt, \quad z \in \mathbb{D}$$

it is clear that $T \in \mathcal{L}(A^p_\alpha)$ if and only if $H \in \mathcal{L}(A^p_\alpha)$ and by [5] this happens if and only if $p > \alpha + 2 > 1$. It also follows that

$$T(f)(z) = \sum_{n\geq 0} \left(\sum_{k \geq 0} \frac{a_k(1+(-1)^{k+n})}{1+k+n} \right) z^n.$$

For more information on weighted Bergman spaces we refer the reader to the monograph [9]. Finally, some more useful notations. If $M \subset \mathbb{C}$ and $c \in \mathbb{C}$, then $cM := \{cm : m \in M\}$. Moreover,

$$\chi_M(z) = \begin{cases} 1, & z \in M; \\ 0, & z \in \mathbb{C} \setminus M; \end{cases} \qquad z \in \mathbb{C}.$$

We also define $\mathbb{D}_{\Re>0} = \mathbb{D} \cap \{z \in \mathbb{C} : \Re z > 0\}$, where $>$ could be replaced by other inequalities. Also, for functions A, B, the relation $A(t) \lesssim B(t)$ means that there is a constant C such that $CA(t) \leq B(t)$ for all t in some explicitly mentioned set. If C is not universal, dependencies will be written as subscripts.

3. Improved norm estimates for the Hilbert matrix operator

Before we proceed, we have the following approximate evaluation type lemma:

Lemma 3.1. *For $p \geq 1$ and $\alpha > -1$, let (f_n) be a sequence of functions with unit A^p_α-norm satisfying: For all $\epsilon > 0$ it holds that $\lim_n \sup_{z \in \mathbb{D} \setminus B(1,\epsilon)} f_n(z) = 0$. Given any function g, with existing limit at 1 from within the disk and such that $f_n g \in A^p_\alpha$ for every n, it holds that*

$$\lim_n \|f_n g\|_{A^p_\alpha} = |g(1)|.$$

The version of this lemma that we will use is the following

Lemma 3.2. *Let $p > \alpha + 2 > 1$ and let $g \in A^p_\alpha$ for which the limit exists at 1 and -1 when taken from within the disk. It holds that*

$$\lim_{c \to \frac{2+\alpha}{p}} \|f_{c,\theta} g\|_{A^p_\alpha} = \left(\frac{\theta^p}{\theta^p + (1-\theta)^p} |g(-1)|^p + \frac{(1-\theta)^p}{\theta^p + (1-\theta)^p} |g(1)|^p \right)^{\frac{1}{p}}.$$

Proof. Note that $f_{c,0}$ and $z \mapsto f_{c,1}(-z)$ satisfies the given conditions in 3.1 when the limit with respect to c is changed to the appropriate limit with respect to n. Moreover, $\left\|\hat{f}_{c,0}\right\|_{A^p_\alpha} = \left\|\hat{f}_{c,1}\right\|_{A^p_\alpha}$, $\hat{f}_{c,\theta} = \theta \hat{f}_{c,1} + (1-\theta)\hat{f}_{c,0}$ and

$$\left\|\hat{f}_{c,\theta}\right\|^p_{A^p_\alpha} = \left\|\chi_{\mathbb{D}_{\Re \leq 0}} \hat{f}_{c,\theta}\right\|^p_{A^p_\alpha} + \left\|\chi_{\mathbb{D}_{\Re > 0}} \hat{f}_{c,\theta}\right\|^p_{A^p_\alpha}.$$

The statement follows from

$$\begin{aligned}
\|f_{c,\theta} g\|^p_{A^p_\alpha} &= \left\|\chi_{\mathbb{D}_{\Re \leq 0}} f_{c,\theta} g\right\|^p_{A^p_\alpha} + \left\|\chi_{\mathbb{D}_{\Re > 0}} f_{c,\theta} g\right\|^p_{A^p_\alpha} \\
&= \frac{\left\|\chi_{\mathbb{D}_{\Re \leq 0}} \hat{f}_{c,\theta} g\right\|^p_{A^p_\alpha}}{\left\|\theta \hat{f}_{c,1}\right\|^p_{A^p_\alpha}} \frac{\left\|\theta \hat{f}_{c,1}\right\|^p_{A^p_\alpha}}{\left\|\hat{f}_{c,\theta}\right\|^p_{A^p_\alpha}} + \frac{\left\|\chi_{\mathbb{D}_{\Re > 0}} \hat{f}_{c,\theta} g\right\|^p_{A^p_\alpha}}{\left\|(1-\theta)\hat{f}_{c,0}\right\|^p_{A^p_\alpha}} \frac{\left\|(1-\theta)\hat{f}_{c,0}\right\|^p_{A^p_\alpha}}{\left\|\hat{f}_{c,\theta}\right\|^p_{A^p_\alpha}}
\end{aligned}$$

and letting $c \to \frac{2+\alpha}{p}$. □

Remark 3.3. The demand on existence of limit from inside the disk can be reduced to only demanding nontangential limits. This is due to the function $z \mapsto |1-z|^{-c}$ having height curves that are circles viewed from the mass/height concentration point 1.

If we instead of $f_{c,\theta}$ consider the normalized version of $z \mapsto \theta\left(\frac{1-z}{2}\right)^n + (1-\theta)\left(\frac{1+z}{2}\right)^n$, the nontangential limits of g give no information about the limit in the lemma. In fact, if the tangential limits of g at ± 1 are zero, the limit in Lemma 3.2 is also zero.

Next, we present one of the crucial results, in order to obtain Theorem 1.1.

Theorem 3.4. *For $4 + 2\alpha \geq p > \alpha + 2 > 1$,*

$$\sup_{f \in B_{A^p_\alpha}} \int_{-1}^{1} \left(\int_{\mathbb{D}} \frac{|f(S_t(z))|^p}{|1-zt|^p} \, dA_\alpha(z) \right)^{\frac{1}{p}} dt = \int_{-1}^{1} \frac{\left(\frac{1}{2}\left[\frac{1}{(1+t)^{4+2\alpha-p}} + \frac{1}{(1-t)^{4+2\alpha-p}}\right] \right)^{\frac{1}{p}}}{(1-t^2)^{1-\frac{2+\alpha}{p}}} \, dt.$$

Proof. The upper bound
By substituting $z \mapsto S_t^{-1}(z)$, we have for every $-1 < t < 1$

$$
\begin{aligned}
\left(\int_{\mathbb{D}} \frac{|f(S_t(z))|^p}{|1-zt|^p}\,dA_\alpha(z)\right)^{\frac{1}{p}} &= \left(\int_{\mathbb{D}} \frac{|f(z)|^p\,|1+tz|^p}{(1-t^2)^p} \frac{(1-t^2)^2}{|1+zt|^4} \frac{(1-t^2)^\alpha}{|1+zt|^{2\alpha}}\,dA_\alpha(z)\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{D}} |f(z)|^p \frac{(1-t^2)^{2+\alpha-p}}{|1+zt|^{4+2\alpha-p}}\,dA_\alpha(z)\right)^{\frac{1}{p}} \qquad (3.1) \\
&\le (1-t^2)^{\frac{2+\alpha}{p}-1}\left(\int_{\mathbb{D}} \frac{|f(z)|^p}{|1+\Re zt|^{4+2\alpha-p}}\,dA_\alpha(z)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Using

$$
g(x) := \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} |f(x+iy)|^p \frac{(1+\alpha)}{\pi}(1-x^2-y^2)^\alpha\,dy,
$$

we have

$$
\int_{-1}^{1}\left(\int_{\mathbb{D}} \frac{|f(S_t(z))|^p}{|1-zt|^p}\,dA_\alpha(z)\right)^{\frac{1}{p}} dt \le \int_{-1}^{1}(1-t^2)^{\frac{2+\alpha}{p}-1}\left(\int_{-1}^{1} \frac{g(x)\,dx}{|1+xt|^{4+2\alpha-p}}\right)^{\frac{1}{p}} dt.
$$

It is clear that $g \ge 0$ (is continuous) and

$$
\int_{-1}^{1} g(x)\,dx = \|f\|^p_{A^p_\alpha},
$$

and hence,

$$
\sup_{f\in B_{A^p_\alpha}} \int_{-1}^{1}\left(\int_{\mathbb{D}} \frac{|f(S_t(z))|^p}{|1-zt|^p}\,dA_\alpha(z)\right)^{\frac{1}{p}} dt \le \sup_{g\in B_{L^1(]-1,1[)}} \int_{-1}^{1} \frac{\left(\int_{-1}^{1} \frac{|g(x)|\,dx}{(1+xt)^{4+2\alpha-p}}\right)^{\frac{1}{p}}}{(1-t^2)^{1-\frac{2+\alpha}{p}}}\,dt.
$$

Every real valued function g with domain $]-1,1[$ can be uniquely written as the sum of an even function, g_e, and an odd function, g_o. For a given $g \in L^1$, let $|g| = g_e + g_o$ and note that $g_e \ge |g_o|$. Define

$$
G_e(t) := \int_{-1}^{1} \frac{g_e(x)\,dx}{(1+xt)^{4+2\alpha-p}} \quad \text{and} \quad G_o(t) := \int_{-1}^{1} \frac{g_o(x)\,dx}{(1+xt)^{4+2\alpha-p}}.
$$

Notice that $G_e(t)$ and $G_o(t)$ are even and odd, respectively, and that $G_e(t) \ge G_o(t)$. It follows that

$$
\begin{aligned}
\int_{-1}^{1}(1-t^2)^{\frac{2+\alpha}{p}-1}\left(\int_{-1}^{1} \frac{|g(x)|\,dx}{(1+xt)^{4+2\alpha-p}}\right)^{\frac{1}{p}} dt &= \int_{-1}^{1}(1-t^2)^{\frac{2+\alpha}{p}-1}\Big(G_e(t)+G_o(t)\Big)^{\frac{1}{p}} dt \\
= \int_{0}^{1}(1-t^2)^{\frac{2+\alpha}{p}-1}\left[\Big(G_e(t)+G_o(t)\Big)^{\frac{1}{p}} + \Big(G_e(t)-G_o(t)\Big)^{\frac{1}{p}}\right] dt.
\end{aligned}
$$

Using the fact that

$$2x^\gamma \geq (x+y)^\gamma + (x-y)^\gamma,\ x \geq y \geq 0, \gamma \in]0,1],$$

we obtain

$$\int_{-1}^{1} (1-t^2)^{\frac{2+\alpha}{p}-1} \left(\int_{-1}^{1} \frac{|g|\ dx}{(1+xt)^{4+2\alpha-p}} \right)^{\frac{1}{p}} dt \leq 2 \int_{0}^{1} (1-t^2)^{\frac{2+\alpha}{p}-1} G_e(t)^{\frac{1}{p}}\, dt.$$

with equality if and only if $g_o \equiv 0$. We have now obtained

$$\sup_{f \in B_{A^p_\alpha}} \int_{-1}^{1} \left(\int_{\mathbb{D}} \frac{|f(S_t(z))|^p}{|1-zt|^p}\, dA_\alpha(z) \right)^{\frac{1}{p}} dt \leq \sup_{g \in B_{L^1_e(]-1,1[)}} \int_{-1}^{1} (1-t^2)^{\frac{2+\alpha}{p}-1} G_e(t)^{\frac{1}{p}}\, dt, \tag{3.2}$$

where $L^1_e(]-1,1[)$ is the subspace of $L^1(]-1,1[)$ consisting of nonnegative, even functions.

Next, we examine the function $G_e(t)$. Using the fact that g is a nonnegative, even function and

$$(1+tx)^{-\gamma} + (1-tx)^{-\gamma} \leq (1+t)^{-\gamma} + (1-t)^{-\gamma},\ x,t \in]-1,1[,\ \gamma \geq 0,$$

we obtain

$$\begin{aligned} G_e(t) &= \int_{0}^{1} g(x) \left[\frac{1}{(1+xt)^{4+2\alpha-p}} + \frac{1}{(1-xt)^{4+2\alpha-p}} \right] dx \\ &\leq \frac{\|g\|_{L^1}}{2} \left[\frac{1}{(1+t)^{4+2\alpha-p}} + \frac{1}{(1-t)^{4+2\alpha-p}} \right]. \end{aligned}$$

This together with (3.2) provides the upper bound.

The lower bound

By (3.1), we have

$$\left(\int_{\mathbb{D}} \frac{|f_{c,\theta}(S_t(z))|^p}{|1-zt|^p}\, dA_\alpha(z) \right)^{\frac{1}{p}} = \left(\int_{\mathbb{D}} |f_{c,\theta}(z)|^p \frac{(1-t^2)^{2+\alpha-p}}{|1+zt|^{4+2\alpha-p}}\, dA_\alpha(z) \right)^{\frac{1}{p}}.$$

Next, define

$$h_{p,\alpha}(t) := \frac{(1-t^2)^{\frac{2+\alpha}{p}-1}}{(1-|t|)^{2\frac{2+\alpha}{p}-1}},\ t \in]-1,1[$$

and note that $h_{p,\alpha}(t) \in L^1$ and

$$\left(\int_{\mathbb{D}} |f_{c,\theta}(z)|^p \frac{(1-t^2)^{2+\alpha-p}}{|1+zt|^{4+2\alpha-p}}\, dA_\alpha(z) \right)^{\frac{1}{p}} \leq h_{p,\alpha}(t).$$

Therefore, by dominated convergence and Lemma 3.2, we obtain

$$\lim_{c \to \frac{2+\alpha}{p}} \int_{-1}^{1} \left(\int_{\mathbb{D}} \frac{|f_{c,\theta}(S_t(z))|^p}{|1-zt|^p}\, dA_\alpha(z) \right)^{\frac{1}{p}} dt = \int_{-1}^{1} \frac{\left(\frac{\theta'}{(1+t)^{4+2\alpha-p}} + \frac{1-\theta'}{(1-t)^{4+2\alpha-p}} \right)^{\frac{1}{p}}}{(1-t^2)^{1-\frac{2+\alpha}{p}}}\, dt, \tag{3.3}$$

where $\theta' = \theta^p/(\theta^p + (1-\theta)^p)$. Letting $\theta = \frac{1}{2}$, we obtain the statement of the theorem. □

Remark 3.5. If we put $\theta \in \{0,1\}$ in (3.3), the right-hand side is

$$\int_{-1}^{1} \frac{(1-t^2)^{\frac{2+\alpha}{p}-1}}{(1-t)^{2\frac{2+\alpha}{p}-1}}\,dt = \int_{0}^{1} \frac{t^{\frac{2+\alpha}{p}-1}}{(1-t)^{\frac{2+\alpha}{p}}}\,dt = \frac{\pi}{\sin\left(\pi\frac{2+\alpha}{p}\right)}, \tag{3.4}$$

where the first equality is the substitution $t \mapsto 2t-1$ and the second is Euler's reflection formula. If $\theta = 1$, the one can use the substitution $t \mapsto -t$ to obtain the left-hand side in (3.4) from the right-hand side in (3.3).

Choosing $\theta = \frac{1}{2}$ in (3.3) maximizes the right-hand side. The expression is in fact increasing w.r.t. $\theta \in]0, \frac{1}{2}[$ and symmetric around $\theta = \frac{1}{2}$. This follows from the simple fact that

$$\begin{aligned}\int_{-1}^{1} \frac{\left(\frac{\theta}{(1+t)^{4+2\alpha-p}} + \frac{1-\theta}{(1-t)^{4+2\alpha-p}}\right)^{\frac{1}{p}}}{(1-t^2)^{1-\frac{2+\alpha}{p}}}\,dt &= \int_{0}^{1} \left(\theta\frac{(1-t)^{2+\alpha-p}}{(1+t)^{2+\alpha-p}} + (1-\theta)\frac{(1+t)^{2+\alpha-p}}{(1-t)^{2+\alpha-p}}\right)^{\frac{1}{p}} \\ &\quad + \left(\theta\frac{(1+t)^{2+\alpha-p}}{(1-t)^{2+\alpha-p}} + (1-\theta)\frac{(1-t)^{2+\alpha-p}}{(1+t)^{2+\alpha-p}}\right)^{\frac{1}{p}} dt\end{aligned}$$

and by differentiation

$$(\theta A + (1-\theta)B)^\gamma + (\theta B + (1-\theta)A)^\gamma$$

is increasing w.r.t. θ on $]0, \frac{1}{2}[$ whenever $\gamma \in]0,1[,\ A, B \geq 0$.

Finally, on the one hand we have by Jensen's inequality

$$\left(\frac{1}{2}\left[\frac{1}{(1+t)^{4+2\alpha-p}} + \frac{1}{(1-t)^{4+2\alpha-p}}\right]\right)^{\frac{1}{p}} > \frac{1}{2}\left(\frac{1}{(1+t)^{\frac{2(2+\alpha)}{p}}} + \frac{1}{(1-t)^{\frac{2(2+\alpha)}{p}}}\right).$$

On the other hand, since $(x+y)^\gamma < x^\gamma + y^\gamma$ for $x, y > 0$ and $\gamma \in]0,1[$, we have

$$\begin{aligned}\left(\frac{1}{2}\left[\frac{1}{(1+t)^{4+2\alpha-p}} + \frac{1}{(1-t)^{4+2\alpha-p}}\right]\right)^{\frac{1}{p}} &< \frac{1}{2^{\frac{1}{p}}}\left(\frac{1}{(1+t)^{\frac{2(2+\alpha)}{p}}} + \frac{1}{(1-t)^{\frac{2(2+\alpha)}{p}}}\right) \\ &= 2^{1-\frac{1}{p}}\frac{1}{2}\left(\frac{1}{(1+t)^{\frac{2(2+\alpha)}{p}}} + \frac{1}{(1-t)^{\frac{2(2+\alpha)}{p}}}\right).\end{aligned}$$

Using $\theta' = \theta^p/(\theta^p + (1-\theta)^p)$, (3.4) yields

$$\int_{-1}^{1} \frac{\left(\frac{\theta'}{(1+t)^{4+2\alpha-p}} + \frac{1-\theta'}{(1-t)^{4+2\alpha-p}}\right)^{\frac{1}{p}}}{(1-t^2)^{1-\frac{2+\alpha}{p}}}\,dt \in \frac{\pi}{\sin\left(\pi\frac{2+\alpha}{p}\right)}\left[1, 2^{1-\frac{1}{p}}\right[\tag{3.5}$$

for any $\theta \in [0,1]$ with the lower bound obtained when $\theta \in \{0,1\}$ and the upper bound obtained when $\theta = \frac{1}{2}$.

We have now obtained an upper bound for the Hilbert matrix operator on $A^p_\alpha,\ p > 2+\alpha > 1$ and we proceed with a proof of Theorem 1.1.

Proof of Theorem 1.1. We have

$$|H(f)(z)| \leq \int_0^1 \left|\frac{f(t)}{1-tz}\right| dt \leq \int_{-1}^1 \left|\frac{f(t)}{1-tz}\right| dt = \int_{-1}^1 \frac{|f(S_t(z))|}{|1-tz|}\, dt.$$

Applying Minkowski's inequality and Theorem 3.4, we obtain

$$\begin{aligned}\|H\|_{\mathcal{L}(A^p_\alpha)} &\leq \sup_{f\in B_{A^p_\alpha}} \left(\int_{\mathbb{D}} \left|\int_{-1}^1 \frac{|f(S_t(z))|}{|1-tz|}\, dt\right|^p dA_\alpha(z)\right)^{\frac{1}{p}} \leq \sup_{f\in B_{A^p_\alpha}} \int_{-1}^1 \left(\int_{\mathbb{D}} \frac{|f(S_t(z))|^p}{|1-zt|^p}\, dA_\alpha(z)\right)^{\frac{1}{p}} dt \\ &= \int_{-1}^1 \frac{\left(\frac{1}{2}\left[\frac{1}{(1+t)^{4+2\alpha-p}} + \frac{1}{(1-t)^{4+2\alpha-p}}\right]\right)^{\frac{1}{p}}}{(1-t^2)^{1-\frac{2+\alpha}{p}}}\, dt.\end{aligned}$$

The rest of the proof follows from (3.5) in Remark 3.5. □

Remark 3.6. Theorem 1.1 yields an improved upper bound for $\|H\|_{\mathcal{L}(A^p_\alpha)}$ when $-1 < \alpha < 0$ and $\alpha + 2 < p < 2(2+\alpha)$. The previously proved bounds, which are given in [12, Theorem 1.3 (ii)] and [2, Theorem 1.1 (ii)] are

$$2^{\frac{2+\alpha}{p}}\left(1+2^{2\frac{2+\alpha}{p}-1}\right)\frac{\pi}{\sin\left(\frac{(2+\alpha)\pi}{p}\right)} \quad \text{and} \quad 2^{\frac{1-\alpha}{p}}\left(1+2^{2\frac{2+\alpha}{p}-1}\right)\frac{\pi}{\sin\left(\frac{(2+\alpha)\pi}{p}\right)},$$

respectively. To see that Theorem 1.1 is an improvement, it suffices to notice that the constants in front of $\pi/\sin((2+\alpha)\pi/p)$ are decreasing w.r.t. p, while $2^{1-\frac{1}{p}}$ is increasing. Comparing the constants when $p = 2(2+\alpha)$ yield the statement. It is also worth noticing that $\lim_{p\to 1} 2^{1-\frac{1}{p}} = 1$, so in some limit sense the new bound is sharp, which is to be expected when comparing Minkowski's inequality with Fubini-Tonelli's theorem. The bound given in Theorem 1.1 is also the smallest known upper bound when $p = \alpha + M$ for any fixed $M > \frac{5}{2}$ and $\alpha > 0$ is large enough (depending on M). In [11, Theorem 1.2 (iii)] the explicit bound for $2+\alpha < p < 2\alpha+3$ is given by

$$\left(1+2^{2\frac{2+\alpha}{p}-1}\right)\frac{\pi}{\sin\left(\frac{(2+\alpha)\pi}{p}\right)}.$$

For such (p,α), we have $p < 2(2+\alpha)$, and hence,

$$\left(1+2^{2\frac{2+\alpha}{p}-1}\right) \geq 2 = \lim_{p\to\infty} 2^{1-\frac{1}{p}} \geq 2^{1-\frac{1}{p}},$$

which proves Theorem 1.1 is an improvement of the result in [11, Theorem 1.2 (iii)], which is the best explicit upper bound found, except for the pairs (p,α) for which Conjecture 1 has been proved (the relevant results are contained in [12,7] and [4]). We will prove that the conjecture has not been proved when $p = \alpha + M$ for any fixed $M > \frac{5}{2}$ and $\alpha > 0$ large enough (depending on M). For such (p,α) we can assume $2+\alpha < p < 2\alpha+3$ holds. Moreover, considering the asymptotics of the bounds of p w.r.t. α for which the conjecture has been proved, it is easy to see that as long as $\alpha > 0$ is large enough, we obtain some new results, except for if the pair (p,α) is contained in the assumption of [12, Theorem 1.2] or [4, Theorem 3.2 (b)]. However, considering the left-hand side of the extra condition given in [12, Theorem 1.2] and more precisely, the function

$$\xi_{p,\alpha}(t) = \int_{(\frac{t}{2-t})^2}^{1} x^{\frac{p}{2}-\alpha-2}(1-x)^{\alpha}\, dx,$$

we see that the exponent $\frac{p}{2} - \alpha - 2 = \frac{\alpha+M}{2} - \alpha - 2$ can be made arbitrarily small by choosing $\alpha > 0$ large enough, making the left-hand side of the extra condition diverge while the right-hand side remains finite. Concerning [4, Theorem 3.2 (b)], it is enough to see that by e.g. Stirling's formula, $\beta(1+\alpha, 2+\frac{\alpha}{2})$ tends exponentially to zero as $\alpha \to \infty$, while the other factors tend at most linearly to either 0 or ∞. The assumption $M > \frac{5}{2}$ assures that the assumptions of [4, Theorem 3.8] are not fulfilled.

Remark 3.7. The left-hand side of the expression in Theorem 3.4 is exactly the result of Minkowski's inequality applied to $\sup_{f\in B_{A^p_\alpha}} \|Tf\|_{A^p_\alpha}$. More generally, assume that $K\colon \mathbb{D}\times]-1,1[\to \mathbb{C}$ is such that

$$g_{c,t,\theta}(z) := \frac{f_{c,\theta}(S_t(z))}{f_{c,\theta}(z)} K(z,t), \quad z \in \mathbb{D}$$

is dominated by a function $g\colon]-1,1[\to \mathbb{R}$ in L^p, that is, $|g_{c,t,\theta}(z)| \le g(t)$ for $c < (2+\alpha)/p$, $\theta \in [0,1]$, $z \in \mathbb{D}$ and $t \in]-1,1[$. Put $c_0 = (2+\alpha)/p$ and $\theta' = \theta^p/(\theta^p + (1-\theta)^p)$ and note that by Minkowski's inequality, we have

$$\left\| \int_{-1}^{1} f_{c,\theta} g_{c,t,\theta}\, dt \right\|_{A^p_\alpha} \le \int_{-1}^{1} \|f_{c,\theta} g_{c,t,\theta}\|_{A^p_\alpha}\, dt.$$

Under some reasonable assumptions (a concrete example is given at the end of this remark), we have by dominated convergence

$$\lim_{c\to c_0} \int_{-1}^{1} \|f_{c,\theta} g_{c,t,\theta}\|_{A^p_\alpha}\, dt = \int_{-1}^{1} \left(\theta' |g_{c_0,t,\theta}(-1)|^p + (1-\theta') |g_{c_0,t,\theta}(1)|^p\right)^{\frac{1}{p}} dt \tag{3.6}$$

and

$$\lim_{c\to c_0} \left\| \int_{-1}^{1} f_{c,\theta} g_{c,t,\theta}\, dt \right\|_{A^p_\alpha} = \theta' \int_{-1}^{1} g_{c_0,t,\theta}(-1)\, dt + (1-\theta') \int_{-1}^{1} g_{c_0,t,\theta}(1)\, dt. \tag{3.7}$$

The right-hand sides of (3.6) and (3.7) are equal iff $g_{c_0,t,\theta}(-1) = g_{c_0,t,\theta}(1)$ or $\theta \in \{0,1\}$ by Jensen's inequality, because $p > 1$. Minkowski's inequality is, therefore, most likely too rough of an estimate to be applied in this manner in order to obtain the exact value of the norm of T. Indeed, compare (3.6) with the proof of the lower bound in Theorem 3.4, and (3.7) with Theorem 3.8. The values of the limits can be compared using (3.5) in Remark 3.5.

In a response to the remark above, we state the following conjecture:

Conjecture 2. For $1 < 2 + \alpha < p$,

$$\|T\|_{\mathcal{L}(A^p_\alpha)} = \frac{\pi}{\sin\left(\pi \frac{2+\alpha}{p}\right)}.$$

The equality in the following theorem justifies Conjecture 2:

Theorem 3.8. *For $p > \alpha + 2 > 1$, and any $\theta \in [0,1]$,*

$$\|T\|_{\mathcal{L}(A^p_\alpha)} \geq \lim_{c \to \frac{2+\alpha}{p}} \|Tf_{c,\theta}\|_{A^p_\alpha} = \frac{\pi}{\sin\left(\pi \frac{2+\alpha}{p}\right)}.$$

Before we prove the theorem, we state the following useful lemma:

Lemma 3.9. *If $\gamma \in]0,1[$ and $\delta < 1 - \max\{\gamma, 1-\gamma\}$, then*

$$\int_{-1}^{1} \sup_{z \in \mathbb{D}} \left| \frac{(1-zt)^{\gamma-1}(1-z^2)^\gamma}{(1-|t|)^{\gamma+\delta}} \right| dt \leq 6 \int_0^1 \frac{dt}{(1-t)^{\max\{\gamma,1-\gamma\}+\delta}}.$$

Proof. First,

$$\int_{-1}^{1} \sup_{z \in \mathbb{D}} \left| \frac{(1-zt)^{\gamma-1}(1-z^2)^\gamma}{(1-|t|)^{\gamma+\delta}} \right| dt = 2 \int_0^1 \sup_{z \in \mathbb{D}} \left| \frac{(1-zt)^{\gamma-1}(1-z^2)^\gamma}{(1-t)^{\gamma+\delta}} \right| dt.$$

Let $t > 0$. We have

$$\sup_{z \in \mathbb{D}_{\Re \leq 0}} \left| \frac{(1-zt)^{\gamma-1}(1-z^2)^\gamma}{(1-t)^{\gamma+\delta}} \right| dt \leq \frac{2}{(1-t)^{\gamma+\delta}}.$$

With the aid of some pictures, we obtain that for $z \in \mathbb{D}_{\Re > 0}$

$$\sup_{t \in]0,1[} \frac{|1-z|}{|1-zt|} \leq \sup_{t \in]0,1[} \frac{|1-(z/|z|)|}{|1-t(z/|z|)|} \leq \frac{2\sin\left(\frac{\arg z}{2}\right)}{\sin(\arg z)} = \left(\cos\left(\frac{\arg z}{2}\right)\right)^{-1} \leq \sqrt{2},$$

which yields that for $t \in]0,1[$ we have

$$\sup_{z \in \mathbb{D}_{\Re > 0}} \left| \frac{(1-zt)^{\gamma-1}(1-z^2)^\gamma}{(1-t)^{\gamma+\delta}} \right| dt \leq 2^\gamma \sup_{z \in \mathbb{D}_{\Re > 0}} \left| \frac{1-z}{1-zt} \right|^\gamma \frac{|1-zt|^{2\gamma-1}}{(1-t)^{\gamma+\delta}} dt \leq \frac{2\sqrt{2}}{(1-t)^{\max\{\gamma,1-\gamma\}+\delta}}.$$

We can conclude that

$$\int_{-1}^{1} \sup_{z \in \mathbb{D}} \left| \frac{(1-zt)^{\gamma-1}(1-z^2)^\gamma}{(1-|t|)^{\gamma+\delta}} \right| \leq 6 \int_0^1 \frac{dt}{(1-t)^{\max\{\gamma,1-\gamma\}+\delta}}. \quad \square$$

Proof of Theorem 3.8. We have

$$\begin{aligned}\hat{f}_{c,\theta}(S_t(z)) &= \theta \frac{(1-zt)^c}{(1-t)^c(1+z)^c} + (1-\theta)\frac{(1-zt)^c}{(1+t)^c(1-z)^c} \\ &= \frac{(1-zt)^c(\theta(1+t)^c(1-z)^c + (1-\theta)(1-t)^c(1+z)^c)}{(1-t^2)^c(1-z^2)^c} \\ &= \hat{f}_{c,\theta}(z)\frac{(1-zt)^c}{(1-t^2)^c}\frac{\theta(1+t)^c(1-z)^c + (1-\theta)(1-t)^c(1+z)^c}{\theta(1-z)^c + (1-\theta)(1+z)^c}.\end{aligned}$$

Define

$$g_{c,t,\theta}\colon z\mapsto \frac{(1-zt)^{c-1}}{(1-t^2)^c}\frac{\theta(1+t)^c(1-z)^c+(1-\theta)(1-t)^c(1+z)^c}{\theta(1-z)^c+(1-\theta)(1+z)^c}$$

and $g_{t,\theta}=g_{(2+\alpha)/p,t,\theta}$.

The branch cuts of $(1\pm z)^c$ are chosen to lie outside of $\mathbb{D}$. We want to compute

$$\lim_{c\to\frac{2+\alpha}{p}}\|Tf_{c,\theta}\|_{A^p_\alpha}=\lim_{c\to\frac{2+\alpha}{p}}\left\|f_{c,\theta}\int_0^1 g_{c,t,\theta}\,dt\right\|_{A^p_\alpha}.$$

It is easy to see that for fixed $\theta\in[0,1]$, $t\in]-1,1[$ and $0<c\le(2+\alpha)/p$ the limits $\lim_{z\to\pm1}g_{c,t,\theta}(z)$ exist from within the disk, more precisely,

$$\forall\epsilon>0\,\exists\delta>0:|1-z|<\delta\text{ and }z\in\mathbb{D}\implies|g_{c,t,\theta}(z)-g_{c,t,\theta}(1)|<\epsilon$$

and

$$\forall\epsilon>0\,\exists\delta>0:|-1-z|<\delta\text{ and }z\in\mathbb{D}\implies|g_{c,t,\theta}(z)-g_{c,t,\theta}(-1)|<\epsilon.$$

First, we assume $\theta\in]0,1[$. Then for $0<c\le(2+\alpha)/p$ it holds that $\inf_{z\in\mathbb{D}}\Re((1\pm z)^c)\ge0$ and since $z\mapsto z^c$ maps $\{z\in\mathbb{C}:1\le|z|<2\text{ and }|\arg z|<\frac{\pi}{4}\}$ into itself, we have

$$\inf_{z\in\mathbb{D}_{\Re\le0}}\Re((1-z)^c)\ge\frac{\sqrt{2}}{2}\quad\text{and}\quad\inf_{z\in\mathbb{D}_{\Re\ge0}}\Re((1+z)^c)\ge\frac{\sqrt{2}}{2}.$$

It follows that

$$|\theta(1-z)^c+(1-\theta)(1+z)^c|\ge\theta\Re((1-z)^c)+(1-\theta)\Re((1+z)^c)\ge\frac{\sqrt{2}}{2}\min\{\theta,1-\theta\}. \tag{3.8}$$

Now, Lemma 3.9 grants the existence of a dominating function $g_\theta(t)\ge|g_{t,\theta}(z)|$, hence, the dominated convergence theorem yields that

$$\lim_{z\to-1}{}^*\left|\int_{-1}^1 g_{t,\theta}(z)\,dt\right|=\int_{-1}^1 g_{t,\theta}(-1)\,dt=\int_{-1}^1\frac{(1+t)^{\frac{2+\alpha}{p}-1}}{(1-t)^{\frac{2+\alpha}{p}}}\,dt$$

and

$$\lim_{z\to1}{}^*\left|\int_{-1}^1 g_{t,\theta}(z)\,dt\right|=\int_{-1}^1\frac{(1-t)^{\frac{2+\alpha}{p}-1}}{(1+t)^{\frac{2+\alpha}{p}}}\,dt=\int_{-1}^1\frac{(1+t)^{\frac{2+\alpha}{p}-1}}{(1-t)^{\frac{2+\alpha}{p}}}\,dt,$$

where $\lim^*$ are limits taken within the disk $\mathbb{D}$. Since the limits exists and $|g_{t,\theta}|\in L^\infty$, we can apply Lemma 3.2 to obtain

$$\begin{aligned}\lim_{c\to\frac{2+\alpha}{p}}\left\|f_{c,\theta}\int_{-1}^1 g_{t,\theta}\,dt\right\|_{A^p_\alpha}&=\theta\lim_{z\to-1}{}^*\left|\int_{-1}^1 g_{t,\theta}(z)\,dt\right|+(1-\theta)\lim_{z\to1}{}^*\left|\int_{-1}^1 g_{t,\theta}(z)\,dt\right|\\&=\int_{-1}^1\frac{(1+t)^{\frac{2+\alpha}{p}-1}}{(1-t)^{\frac{2+\alpha}{p}}}\,dt=\frac{\pi}{\sin\left(\pi\frac{2+\alpha}{p}\right)}.\end{aligned} \tag{3.9}$$

Since

$$\left| \|Tf_{c,\theta}\|_{A^p_\alpha} - \left\| f_{c,\theta} \int_0^1 g_{t,\theta}\, dt \right\|_{A^p_\alpha} \right| \leq \left\| f_{c,\theta} \int_0^1 (g_{c,t,\theta} - g_{t,\theta}\, dt) \right\|_{A^p_\alpha}, \tag{3.10}$$

it remains to show that

$$\lim_{c \to \frac{2+\alpha}{p}} \left\| f_{c,\theta} \int_0^1 (g_{c,t,\theta} - g_{t,\theta}\, dt) \right\|_{A^p_\alpha} = 0. \tag{3.11}$$

To this end, put $c_0 = (2+\alpha)/p$ and partition $g_{c,t,\theta} = g^{(1)}_{c,t}(z)h^{(1)}_{c,t}(z) + g^{(-1)}_{c,t}(z)h^{(-1)}_{c,t}(z)$, where

$$g^{(1)}_{c,t}(z) = \frac{(1-zt)^{c-1}(1-z)^c}{(1-t)^c}, \quad g^{(-1)}_{c,t}(z) = \frac{(1-zt)^{c-1}(1+z)^c}{(1+t)^c},$$

$$h^{(1)}_{c,t}(z) = \frac{\theta}{\theta(1-z)^c + (1-\theta)(1+z)^c} \quad \text{and} \quad h^{(-1)}_{c,t}(z) = \frac{(1-\theta)}{\theta(1-z)^c + (1-\theta)(1+z)^c}.$$

Now

$$\int_{-1}^1 g^{(1)}_{c,t}h^{(1)}_{c,t} - g^{(1)}_{c_0,t}h^{(1)}_{c_0,t}\, dt = \int_{-1}^1 (g^{(1)}_{c,t} - g^{(1)}_{c_0,t})h^{(1)}_{c,t}\, dt + \int_{-1}^1 g^{(1)}_{c_0,t}(h^{(1)}_{c,t} - h^{(1)}_{c_0,t})\, dt.$$

For the last integral, Lemma 3.9 yields

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \left| \int_{-1}^1 g^{(1)}_{c_0,t}(z)(h^{(1)}_{c,t}(z) - h^{(1)}_{c_0,t}(z))\, dt \right| \\ &\qquad \leq \sup_{z\in\mathbb{D}} \sup_{s\in]-1,1[} \left| h^{(1)}_{c,s}(z) - h^{(1)}_{c_0,s}(z) \right| \int_{-1}^1 \sup_{w\in\mathbb{D}} \left| g^{(1)}_{c_0,t}(w) \right| dt < \infty, \end{aligned} \tag{3.12}$$

and so the right-hand side tends to zero as $c \to c_0$. Moreover, for $z \in \mathbb{D}$ using (3.8)

$$\begin{aligned} \left| \int_{-1}^1 (g^{(1)}_{c,t} - g^{(1)}_{c_0,t})h^{(1)}_{c,t}\, dt \right| &\leq \frac{\sqrt{2}}{\min\{\theta, 1-\theta\}} \int_{-1}^1 \left| g^{(1)}_{c,t} - g^{(1)}_{c_0,t} \right| dt \\ &= \frac{\sqrt{2}}{\min\{\theta, 1-\theta\}} \int_{-1}^1 \left| g^{(1)}_{c,t} \right| \left| 1 - \frac{g^{(1)}_{c_0,t}}{g^{(1)}_{c,t}} \right| dt \end{aligned}$$

and

$$\begin{aligned} \int_{-1}^1 \left| g^{(1)}_{c,t} \right| \left| 1 - \frac{g^{(1)}_{c_0,t}}{g^{(1)}_{c,t}} \right| dt &= \int_{-1}^1 \left| \frac{(1-zt)^{c-1}(1-z)^c}{(1-t)^c} \right| \left| 1 - \frac{(1-zt)^{c_0-c}(1-z)^{c_0-c}}{(1-t)^{c_0-c}} \right| dt \\ &\leq \left(\sup_{w\in\mathbb{D}} \sup_{s\in]-1,1[} \left| (1-s)^{c_0-c} - (1-ws)^{c_0-c}(1-w)^{c_0-c} \right| \right) \int_{-1}^1 \left| \frac{(1-zt)^{c-1}(1-z)^c}{(1-t)^{c+c_0-c}} \right| dt. \end{aligned}$$

Furthermore, for $c \in]\frac{2}{3}c_0, c_0[$, we have by Lemma 3.9,

$$\int_{-1}^{1} \left| \frac{(1-zt)^{c-1}(1-z)^c}{(1-t)^{c+c_0-c}} \right| dt$$

$$\leq \int_{-1}^{0} \left| \frac{(1-zt)^{c-1}(1-z)^c}{(1-t)^{c_0}} \right| dt + (\chi_{\mathbb{D}_{\Re>0}}(z) + \chi_{\mathbb{D}_{\Re\leq 0}}(z)) \int_{0}^{1} \left| \frac{(1-zt)^{c-1}(1-z)^c}{(1-t)^{c+c_0-c}} \right| dt$$

$$\leq \int_{-1}^{0} \frac{2}{(1+t)^{1-c}}\, dt + 6 \int_{0}^{1} \frac{dt}{(1-t)^{\max\{c,1-c\}+c_0-c}} + \int_{0}^{1} \frac{2}{(1-t)^{c_0}}\, dt$$

$$\leq \int_{-1}^{0} \frac{2}{(1+t)^{1-\frac{2}{3}c_0}}\, dt + 8 \int_{0}^{1} \frac{dt}{(1-t)^{\max\{c_0,1-\frac{1}{3}c_0\}}} < \infty.$$

We can, therefore, by the dominated convergence theorem conclude that

$$\lim_{c\to c_0} \left| \int_{-1}^{1} (g_{c,t}^{(1)} - g_{c_0,t}^{(1)}) h_{c,t}^{(1)}\, dt \right| = 0$$

and with (3.12), we obtain

$$\lim_{c\to c_0} \left| \int_{-1}^{1} g_{c,t}^{(1)} h_{c,t}^{(1)} - g_{c_0,t}^{(1)} h_{c_0,t}^{(1)}\, dt \right| = 0.$$

Similar calculations can be done to conclude that

$$\lim_{c\to c_0} \left| \int_{-1}^{1} g_{c,t}^{(-1)} h_{c,t}^{(-1)} - g_{c_0,t}^{(-1)} h_{c_0,t}^{(-1)}\, dt \right| = 0,$$

and hence, (3.11) holds. Combining this with (3.9) and (3.10) yields the lower bound for $\|T\|_{\mathcal{L}(A^p_\alpha)}$. □

4. Essential norm of the Hilbert matrix operator on weighted Bergman spaces

Proof of Theorem 1.2. In [11, Proof of Theorem 1.2] it is stated that

$$\|T_t f\|_{A^p_\alpha} = \psi_{p,\alpha}(t) \left((\alpha+1) \int_{B(c_t,\rho_t)} |w|^{p-4-2\alpha} |f(w)|^p\, g_{t,\alpha}(w)\, dA(w) \right)^{\frac{1}{p}}$$

where

$$c_t = \frac{1}{2-t} \quad \rho_t = \frac{1-t}{2-t} \quad \psi_{p,\alpha}(t) = \frac{t^{\frac{2+\alpha}{p}-1}}{(1-t)^{\frac{2+\alpha}{p}}}$$

and

$$g_{t,\alpha}(w) = \left(\frac{\rho_t^2 - |w - c_t|^2}{\rho_t}\right)^\alpha.$$

As in [13, Section 4] we partition $B(c_t, \rho_t) = D_{>R,t} \cup D_{\le R,t}$, where $D_{\le R,t} = B(c_t, \rho_t) \cap \overline{B(0,R)}$, $D_{>R,t} = B(c_t, \rho_t) \setminus \overline{B(0,R)}$ and $R > \frac{1}{2}$.

On the one hand, we have

$$\psi_{p,\alpha}(t)^p(\alpha+1) \int\limits_{D_{\le R,t}} |w|^{p-4-2\alpha} |f(w)|^p g_{t,\alpha}(w)\, dA(w) \le \|T_t(1)\|_{A_\alpha^p}^p \sup_{|z|\le R} |f(z)|^p$$

and by [9, Theorem 1.7] we have

$$\|T_t(1)\|_{A_\alpha^p}^p \asymp_\alpha \int\limits_{\mathbb{D}} \left|\frac{1}{1-(1-t)z}\right|^p (1-|z|^2)^\alpha dA(z) \lesssim_{p,\alpha} t^{2+\alpha-p},\ t \in]0,1[. \tag{4.1}$$

On the other hand, we have

$$\int\limits_{D_{>R,t}} |w|^{p-4-2\alpha} |f(w)|^p g_{t,\alpha}(w)\, dA(w) \le \max\{1, R^{p-4-2\alpha}\} \int\limits_{D_{>R,t}} |f(w)|^p g_{t,\alpha}(w)\, dA(w),$$

and continuing as in [4, Proof of Theorem 6.5], we obtain

$$\psi_{p,\alpha}(t)^p(\alpha+1) \int\limits_{D_{>R,t}} |w|^{p-4-2\alpha} |f(w)|^p g_{t,\alpha}(w)\, dA(w) \le \psi_{p,\alpha}(t)^p \max\{1, R^{p-4-2\alpha}\} \|f\|_{A_\alpha^p}^p.$$

Together with (4.1), we have now obtained

$$\|Hf\|_{A_\alpha^p} \le \int_0^1 \|T_t f\|_{A_\alpha^p}\, dt \le C_{p,\alpha} \sup_{|z|\le R} |f(z)| + \max\{1, R^{1-\frac{2(2+\alpha)}{p}}\} \|f\|_{A_\alpha^p} \int_0^1 \psi_{p,\alpha}(t)\, dt$$

for some $C_{p,\alpha} > 0$. Let $(L_n) \subset \mathcal{L}(A_\alpha^p)$ be the sequence of compact operators given in [13, Lemma 3.2] (see also [14]). It follows that

$$\lim_{n\to\infty} \|H - HL_n\|_{A_\alpha^p} \le \int_0^1 \psi_{p,\alpha}(t)\, dt = \max\{1, R^{1-\frac{2(2+\alpha)}{p}}\} \frac{\pi}{\sin\left(\pi\frac{2+\alpha}{p}\right)},$$

for all $0 < R < 1$. Let $R \to 1$ to obtain the upper bound for the essential norm.

In [11] a lower bound for the norm was calculated considering the sequence $(f_{c,0})$ as $c \to (2+\alpha)/p$. The sequence converges weakly to zero since it converges to zero on compact subsets of $\mathbb{D}$ and A_α^p is reflexive for $p > 1$. Therefore, $\|K(f_{c,0})\|_{A_\alpha^p} = 0$ and hence,

$$\begin{aligned}\|H\|_{e,\mathcal{L}(A_\alpha^p)} &\ge \inf_K \lim_c \|(H-K)(f_{c,0})\|_{A_\alpha^p} \ge \lim_c \|H(f_{c,0})\|_{A_\alpha^p} - \sup_K \lim_c \|K(f_{c,0})\|_{A_\alpha^p} \\ &= \lim_c \|H(f_{c,0})\|_{A_\alpha^p} = \frac{\pi}{\sin\left(\pi\frac{2+\alpha}{p}\right)},\end{aligned}$$

where $\inf_K$ and $\sup_K$ means infimum and supremum respectively, over compact operators K; the last equality is found in the proof of Theorem 1.1 [11]. □

Remark 4.1. We have now obtained an extension of [13, Corollary 9.4]. Let $p > \alpha + 2 > 1$. If $H\colon A^p_\alpha \to A^p_\alpha$ is not norm attaining, then

$$\|H\|_{\mathcal{L}(A^p_\alpha)} = \frac{\pi}{\sin\left(\pi\frac{2+\alpha}{p}\right)}.$$

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References

[1] Vladimir Božin, Boban Karapetrović, Norm of the Hilbert matrix on Bergman spaces, J. Funct. Anal. 274 (2) (2018) 525–543, MR 3724148.
[2] Dušica Bralović, Boban Karapetrović, New upper bound for the Hilbert matrix norm on negatively indexed weighted Bergman spaces, Bull. Malays. Math. Sci. Soc. 45 (3) (2022) 1183–1193, MR 4412316.
[3] Jineng Dai, Norm of the Hilbert matrix operator on the Korenblum space, J. Math. Anal. Appl. 514 (1) (2022) 126270, MR 4413318.
[4] Jineng Dai, On the norm of the Hilbert matrix operator on weighted Bergman spaces, J. Funct. Anal. 287 (9) (2024) 110587, MR 4779811.
[5] Epameinondas Diamantopoulos, Hilbert matrix on Bergman spaces, Ill. J. Math. 48 (3) (2004) 1067–1078, MR 2114270.
[6] Epameinondas Diamantopoulos, Aristomenis G. Siskakis, Composition operators and the Hilbert matrix, Stud. Math. 140 (2) (2000) 191–198, MR 1784632.
[7] Dušica Dmitrović, Boban Karapetrović, On the Hilbert matrix norm on positively indexed weighted Bergman spaces, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 117 (4) (2023) 138, MR 4609670.
[8] Milutin Dostanić, Miroljub Jevtić, Dragan Vukotić, Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type, J. Funct. Anal. 254 (11) (2008) 2800–2815, MR 2414221.
[9] Haakan Hedenmalm, Boris Korenblum, Kehe Zhu, Theory of Bergman Spaces, Graduate Texts in Mathematics, vol. 199, Springer-Verlag, New York, 2000, MR 1758653.
[10] Miroljub Jevtić, Boban Karapetrović, Hilbert matrix on spaces of Bergman-type, J. Math. Anal. Appl. 453 (1) (2017) 241–254, MR 3641772.
[11] Boban Karapetrović, Norm of the Hilbert matrix operator on the weighted Bergman spaces, Glasg. Math. J. 60 (3) (2018) 513–525, MR 3830533.
[12] Boban Karapetrović, Hilbert matrix and its norm on weighted Bergman spaces, J. Geom. Anal. 31 (6) (2021) 5909–5940, MR 4267631.
[13] Mikael Lindström, Santeri Miihkinen, David Norrbo, Exact essential norm of generalized Hilbert matrix operators on classical analytic function spaces, Adv. Math. 408 (2022) 108598, MR 4460279.
[14] Mikael Lindström, Santeri Miihkinen, David Norrbo, Corrigendum to "Exact essential norm of generalized Hilbert matrix operators on classical analytic function spaces", Adv. Math. 408 (2022) 108598, Adv. Math. 450 (2024) 109759.
[15] Mikael Lindström, Santeri Miihkinen, Niklas Wikman, On the exact value of the norm of the Hilbert matrix operator on weighted Bergman spaces, Ann. Fenn. Math. 46 (1) (2021) 201–224, MR 4277807.
[16] Wilhelm Magnus, On the spectrum of Hilbert's matrix, Am. J. Math. 72 (1950) 699–704, MR 41358.