Time-consistent consumption, investment, and proportional reinsurance in market models with Markovian regime switching

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Time-consistent consumption-investment and proportional reinsurance in market model under Markovian regime-switching

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Abstract. This paper presents a characterization of equilibrium in a game-theoretic description of discounting stochastic consumption, investment and reinsurance problem, in which the controlled state process evolves according to a multi-dimensional linear stochastic differential equation, when the noise is driven by a Brownian motion under the effect of a Markovian regime-switching. The running and the terminal costs in the objective functional, are explicitly depended on some general discount functions, which create the time-inconsistency of the considered model. Open-loop Nash equilibrium controls are described through some necessary and sufficient equilibrium conditions as well as a verification result. A state feedback equilibrium strategy is achieved via certain partial differential-difference equation. As an application, we study an investment-consumption and equilibrium reinsurance/new business strategies for some particular cases of power and logarithmic utility functions. A numerical example is provided to demonstrate the efficacy of theoretical results.

Keys words: Stochastic Optimization, Investment-Consumption Problem, Merton Portfolio Problem, Non-Exponential Discounting, Time-Inconsistency, Equilibrium Strategies, Stochastic Maximum Principle.
MSC 2010 subject classifications, 93E20, 60H30, 93E99, 60H10.

1 Introduction

In recent years, time-inconsistent control problems have received significant attention in economics and mathematical finance. For a dynamic control issue, time-inconsistency suggests that the Bellman's optimality principle does not hold in this case. In another way, a restriction of an optimal control for a definite starting data on a future time interval may not be optimal for that associated initial data. This arises, for example, in mean-variance control issues and utility maximization situations for consumption-investment strategies with non-exponential discounting.

The common assumption in usual discounted investment-consumption problems is that the discount rate is constant over time, this assumption offers the possibility to compare outcomes occurring at various times by discounting future utility by some constant factor. On the other hand, results from experimental studies contradict this assumption, implying that discount rates for the near future are much lower than discount rates for the time further away in the future. Ainslie, in [1], established empirical studies on human and animal behavior and discovered that discount functions are almost hyperbolic, meaning that they decrease like a negative power of time rather than an exponential. According to Loewenstein & Prelec in [34], economic decision makers are impatient about selections in the short term but are more patient when choosing between long-term alternatives, so a hyperbolic discount function would be more realistic. Consequently, when the discount function is non-exponential, discounted utility models become time-inconsistent, that is, they do not satisfy the Bellman's optimality principle, and the classical dynamic programming technique may not be applied to solve these problems.

There are two fundamental methods to handling the time-inconsistency in the non-exponential discounted utility models. In the first one, under the concept of naive agents, every decision is made without considering that their preferences may change in the near future. At any time $t \in [0, T]$, the agent will solve the problem like

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a standard optimal control problem with an initial condition $X(t) = x_t$. If we assume that the naive agent solves the problem at time 0, his or her solution is a so-called pre-commitment solution. In other terms, it is optimal as long as the agent can pre-commit his or her future behavior at time $t = 0$. Kydland & Prescott in [30] argue that a pre-committed strategy may be economically significant in some situations.

The second method consists of the formulation of a time-inconsistent decision problem as a non-cooperative game among incarnations of the decision maker at various instants of time. Nash equilibrium of these strategies is then considered to determine the new notion of the solution to the original problem. Strotz in [47] was the first to apply this game perspective to dealing with the dynamic time-inconsistent decision problem posed by the deterministic Ramsay problem; see [44]. Then, by capturing the concept of non-commitment, by allowing the commitment period being infinitesimally small, he presented a primitive notion of Nash equilibrium strategy. More works along this line has been done in both discrete and continuous time by Pollak [43], Phelps and Pollak [41], Goldman [21], Barro [6] and Krusell & Smith [29]. Using the same game theoretic approach, Ekland & Lazrak [16] and Marín-Solano & Navas [35] considered an optimal investment-consumption problem under non-exponential discount function in the deterministic framework. They described the equilibrium strategies via a value function which should satisfy a certain non linear differential equation with a non local term depends on the global behavior of the solution, called "extended HJB equation". In this case, every decision at time t is made by a $t-$agent which denotes the incarnation of the controller at time t and is called a "sophisticated $t-$agent" in [35].

Björk & Murgoci in [8] extended the notion to the stochastic setting, in which the controlled dynamic is driven by a general class of Markov process and a fairly general objective function. Yong in [51], by a discretization of time, investigated a class of time-inconsistent deterministic linear quadratic models and obtained equilibrium controls through some class of Riccati-Voltera equations. Yong in [52], also by a discretization of time, studied a general discounting time-inconsistent stochastic optimal control problem and described a feedback time-consistent Nash equilibrium control through the "equilibrium HJB equation". In a series of works, Basak & Chabakauri [7], Hu et al. [25], Czichowsky [14] and Björk et al. [9] studied the time-inconsistent mean variance problem.

For the equilibrium strategies in optimal consumption-investment problem under a general discount function, Ekeland & Pirvu [17] were the first to discover the Nash equilibrium strategies in which the price process of the risky asset is driven by geometric Brownian motion. They described the equilibrium strategies via the solutions of a flow of BSDEs and showed that for a special form of the discount function, the BSDEs reduce to a system of two ODEs that has a solution. Ekeland et al. in [18] added life insurance to the investor's portfolio and they used an integral equation to describe the equilibrium strategy. In [52], Yong addressed the problem of time-inconsistent consumption-investment under a power utility function. Following Yong's method. Zhao et al. in [54], investigated the consumption-investment problem under a general discount function and a logarithmic utility function. Furthermore, Zou et al. in [56], studied the equilibrium consumption-investment strategies for Merton's portfolio problem under stochastic hyperbolic discounting.

Recently, Markov regime-switching models have received a lot of attention in financial applications, see e.g., [55], [12], [13], [50] and [32]. Markov regime-switching models allow the market to face shocks at random times. A standard example of such a regime would be a bull market, in which stock prices are generally rising. After a shock, the market's behaviour fundamentally changes. The shock is represented as a switch of regime. Zhou and Yin [55] are the first to investigate the mean-variance optimization problem under a continuous time Markov regime-switching financial market, by using techniques of stochastic linear-quadratic control, they derived the mean-variance efficient portfolios and efficient frontiers based on solutions of two systems of linear ordinary differential equations. Chen et al. [12], Chen and Yang [13] investigated the mean-variance asset-liability management problem in continuous-time and in multi-period settings, respectively. Wei, Wong, Yam and Yung [50] investigated the mean-variance asset-liability management problems under a continuous time Markov regime-switching setting. Following the approach developed in [8], they derived a time consistent investment strategy explicitly. Liang and Song [32] studied optimal investment and reinsurance problems under partial information for insurer with mean-variance utility, where the drift rate of stock and insurer's risk aversion are Markov-modulated.

In this paper, we investigate equilibrium solutions for a non-exponential discounted time-inconsistent investment-consumption and reinsurance problem under continuous time Markov regime switching framework and a general utility function. Different from [35] and [17], in which the authors provided explicit solutions for special forms of the discount factor, the non-exponential discount function in our model is in a fairly general form. Moreover, we consider open-loop equilibrium strategies, as defined in [25] and [26], which differs from the majority of the existing literature on this subject. It's also worth noting that the time-inconsistency, in our paper is due to non-exponential discounting in the objective function, whereas the works [25] and [26] are concerned with a different type of time-inconsistency caused by non-linear terms of expectations in the terminal cost. Moreover, in our paper, the objective functional is not reduced to the quadratic form as in [25] and [26].

We concentrate on a variational technique approach that leads to a version of a necessary and sufficient condition for equilibrium, which includes a flow of forward-backward stochastic differential equations (FBSDEs) and an equilibrium condition. We also provide a verification theorem that covers some possible cases of utility functions. Then, by decoupling the flow of the FBSDEs, we get a closed-loop representation of the equilibrium strategies through a parabolic non-linear partial differential-difference equation (PDDE). We show that for a special form of the utility function (logarithmic and power) the PDDE reduces to a system of ODEs that has an explicit solution. Noting that, Hamaguchi in [23] presented some equilibrium conditions for a general time-inconsistent investment and consumption model in a possibly incomplete market under general discount functions with random endowments. These conditions are connected to the solvability of an equivalent fully coupled FBSDE system, which is more feasible than a flow of FBSDEs studied in [25] and [26].

We emphasize that, different from most of the existing studies on this topic, where feedback equilibrium strategies are obtained via several very difficult non-linear integro-differential equations, in our paper, we derive an explicit representation of the equilibrium strategies via simple ODEs. This technique can also present the necessary and sufficient conditions for characterizing equilibrium strategies, whereas the extended HJB techniques can only provide, in general, the sufficient condition in the form of a verification theorem that describes the equilibrium strategies.

The paper is organized as follows. In Section 2, we formulate the problem and provide the necessary notations and preliminaries. In Section 3 we give the main results of the paper, Theorem 5 and Theorem 8, which characterizes the equilibrium strategies by some necessary and sufficient conditions. In Section 4, we derive an explicit representation of the equilibrium consumption-investment and reinsurance strategies. Section 5 An explicit representation of the equilibrium strategies is derived for a special form of the utility function (logarithmic and power). The paper concludes with an Appendix giving some proofs.

2 Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where $\mathbb{F} := \{\mathcal{F}_t | t \in [0,T]\}$ is a right-continuous, $\mathbb{P}$-completed filtration to which all of the processes outlined below are adapted, such as the Markov chain and the Brownian motions.

The Markov chain $\alpha(\cdot)$ is assumed to take values in finite state space $\chi = \{e_1, e_2, ..., e_D\}$ where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$ and the j-th component of e_i is the Kronecker delta δ_{ij} for each $(i,j) \in \{1,...,D\}^2$. $\mathcal{G} := (g_{ij})_{1\leq i,j\leq D}$ represents the rate matrix of the Markov chain under $\mathbb{P}$. As a result, g_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t, for each $(i,j) \in \{1,...,D\}^2$. Note that for, $i \neq j$, $g_{ij} \geq 0$ and $\sum_{j=1}^{D} g_{ij} = 0$, thus $g_{ii} \leq 0$. In the sequel, for each $i, j = 1, 2, ..., D$ with $i \neq j$, we assume that $g_{ij} > 0$ consequently, $g_{ii} < 0$. We have the following semimartingale representation of the Markov chain $\alpha(\cdot)$ obtained from Elliott et al. [19]

$$\alpha(t) = \alpha(0) + \int_0^t \mathcal{G}^\top \alpha(\tau) d\tau + \mathcal{M}(t),$$

where $\{\mathcal{M}(t) | t \in [0,T]\}$ is an $\mathbb{R}^D$-valued, $(\mathbb{F}, \mathbb{P})$-martingale.

We first provide a set of Markov jump martingales linked with the chain $\alpha(.)$, which will be used to model the controlled state process. For each $(i,j) \in \{1,...,D\}^2$, with $i \neq j$, and $t \in [0,T]$, denote by $J^{ij}(t) := g_{ij} \int_0^t \langle \alpha(\tau-), e_i \rangle d\tau + m_{ij}(t)$ the number of jumps from state e_i to state e_j up to time t, where $m_{ij}(t) := \int_0^t \langle \alpha(\tau-), e_i \rangle \langle d\mathcal{M}(\tau), e_j \rangle d\tau$ an $(\mathbb{F}, \mathbb{P})$-martingale. $\tilde{\Phi}^j(t)$ denotes the number of jumps into state e_j up to time t, for each fixed $j = 1, 2, ..., D$, then

$$\begin{aligned}\tilde{\Phi}^j(t) &= \sum_{i=1, i\neq j}^{D} J^{ij}(t), \\ &= \sum_{i=1, i\neq j}^{D} g_{ij} \int_0^t \langle \alpha(\tau-), e_i \rangle d\tau + \Phi^j(t),\end{aligned}$$

with $\Phi^j(t) := \sum\limits_{i=1, i\neq j}^{D} m_{_{ij}}(t)$ is an $(\mathbb{F},\mathbb{P})$-martingale for each $j = 1, 2, ..., D$. Set for each $j = 1, 2, ..., D$

$$g_{_j}(t) = \sum_{i=1, i\neq j}^{D} g_{ij} \int_0^t \langle \alpha(\tau), e_i \rangle \, d\tau.$$

Then, the process $\Phi^j(t) = \tilde{\Phi}^j(t) - g_{_j}(t)$ is an $(\mathbb{F},\mathbb{P})$-martingale, for each $j = 1, 2, ..., D$.

2.1 Notations

Throughout this paper, we use the following notations:

$M^\top$: the transpose of the vector (or matrix) M, $\langle \chi, \zeta \rangle$: the inner product of χ and ζ, that is, $\langle \chi, \zeta \rangle := tr(\chi^T \zeta)$. For a function f, we denote by f_x (resp. f_{xx}) the first (resp. the second) derivative of f with respect to the variable x.

For any Euclidean space E with Frobenius norm $|\cdot|$ we let, for any $t \in [0, T]$,

- $\mathbb{L}^p(\Omega, \mathcal{F}_t, \mathbb{P}; E)$: for any $p \geq 1$, the set of $E-$valued $\mathcal{F}_t-$measurable random variables X, such that $\mathbb{E}[|X|^p] < \infty$.
- $\mathcal{L}^2_{\mathcal{F}}(t, T; E)$: the space of $E-$valued, $(\mathcal{F}_s)_{s\in[t,T]}$ $-$adapted continuous processes $\mathcal{Y}(\cdot)$, with

$$\|\mathcal{Y}(\cdot)\|_{\mathcal{L}^2_{\mathcal{F}}(t,T;E)} = \sqrt{\mathbb{E}\left[\sup_{s\in[t,T]} |\mathcal{Y}(s)|^2\right]} < \infty.$$

- $\mathcal{M}^p_{\mathcal{F}}(t, T; E)$: for any $p \geq 1$, the space of $E-$valued, $(\mathcal{F}_s)_{s\in[t,T]}$ $-$adapted processes $\mathcal{Z}(\cdot)$, with

$$\|\mathcal{Z}(\cdot)\|_{\mathcal{M}^p_{\mathcal{F}}(t,T;E)} = \mathbb{E}\left[\int_t^T |\mathcal{Z}(s)|^p \, ds\right]^{\frac{1}{p}} < \infty.$$

- $\mathcal{M}^{g,2}_{\mathcal{F},p}(t, T; E)$: the space of $E-$valued, $(\mathcal{F}_s)_{s\in[t,T]}$ $-$predictable processes $\mathcal{X}(\cdot)$, with

$$\|\mathcal{X}(\cdot)\|_{\mathcal{M}^{g,2}_{\mathcal{F},p}(t,T;E)} = \mathbb{E}\left[\int_t^T \sum_{j\neq i} |\mathcal{X}_{ij}(s)|^2 g_{ij}(s) \, ds\right] < \infty.$$

2.2 Risk process

The classical risk process of an insurer is described by the following stochastic differential equation

$$dR_1(s) = cds - d\sum_{i=1}^{L(s)} Y_i \tag{2.1}$$

where the premium rate c is a constant, implying that the insurance company gets deterministically units of money per unit time. Meanwhile, when a claim occurs, the insurance company must pay a stochastic sum of money. Assume that the number of claims throughout the time interval $[0, t]$ is represented by the counting process $\{L(s)\}_{s\geq 0}$, Y_i is the i-th claim size, and $\{Y_i\}_{i\geq 1}$ are i.i.d. random variables that are independent of $L(s)$. We suppose $\{L(s)\}_{s\geq 0}$ is a Poisson process with intensity $\lambda_L > 0$, which means $\mathbb{E}[L(s)] = \lambda_L s$. Y is a generic random variable with the same distribution as $\{Y_i\}_{i\geq 1}$. The first and second moments of Y are $m_Y > 0$ and $\sigma_Y > 0$. The expected value principle is supposed to be used to determine the premium rate c, i.e., $c = (1+\eta_1)\dfrac{\mathbb{E}[L(s)]\mathbb{E}(Y)}{s} = (1+\eta_1)\lambda_L m_Y$ with safety loading $\eta_1 > 0$, then the insurance company gets the expected profit $\mathbb{E}[dX(s)] = (c - \lambda_L m_Y)ds = \eta_1 \lambda_L m_Y ds$. In this paper according to Grandell [22], we consider the diffusion approximation, i.e., approximating the classical risk model by a Brownian motion with drift. This approximation is mathematically based on the theory of weak convergence of probability measures. The way to express this diffusion approximation is that if the classical risk model is viewed as "large deviation", the diffusion model is associated to the "central limit theorem". The diffusion approximation is extensively used in the literature on optimal problems for insurers, such as Browne [11], Bai & Zhang [5], etc. Hence, the diffusion approximation of the classical risk process is as follows

$$dR_2(s) = (1+\eta_1)\lambda_L m_Y ds - \lambda_L m_Y ds + \sqrt{\lambda_L \sigma_Y} dW_0(s), \tag{2.2}$$

where $W_0(\cdot)$ is a standard Brownian motion. For more details on this diffusion approximation, see Grandell [22] (pages $15-17$). Another remarkable work is [38]. An insurance company gets the premium but it will also face the risk of paying claims. If the risk is too dangerous, the insurer may decide to transmit part of the risk to another insurer. Reinsurance is the procedure that transfers risk from one insurance company to another. As the second insurance company is called reinsurer. The reinsurance company frequently does the same thing, i.e., it transfers part of its own risk to a third company, and so on. By transferring on parts of risks, large risks are divided into a number of smaller sections taken up by different risk carriers.This risk exchange procedure reduces the danger of large claims for individual insurers. Reinsurance can take several forms, including proportional reinsurance, excess-loss reinsurance, and stop-loss reinsurance, and so on. In this work, we consider the proportional reinsurance, which is widely used in practice. Let $a(s)$ be the retention level of new business (particularly, reinsurance business) acquired at time s. It means that the insurer pays $a(s)Y$ for the claim Y that occurs at time s, whereas the new businessman (particularly, the reinsurer) pays $(1-a(s))Y$. The reinsurance premium is also supposed to be calculated through the expected value principle, i.e., the premium is to be paid at rate$(1-a(s))c_1 = (1-a(s))(1+\theta)\lambda_L m_Y$ for this business, where $\theta > 0$ is the safety loading of the new businessman, where we assume that $\eta_1$and θ are equal. As a result, the expected profit of the reinsurance company is $\{(1-a(s))c_1 - (1-a(s))\lambda_L m_Y\} ds = (1-a(s))\theta\lambda_L m_Y ds$ in $[s, s+ds)$. Note that for the first insurance company, $a(s) \in [0,1]$ corresponds to a reinsurance cover, $a(s) > 1$ would mean that the company is able to take on additional insurance business from other companies (i.e., operate as a reinsurer for other cedents) and $a(s) < 0$ indicates other new businesses. The following SDE describes the reserve process with new business before investment. In order to emphasise the dependence on the reinsurance price, we let the safety coefficient of the reinsurer depend on the current regime by letting all other variables unchanged. Thus, instead of (2.2), we consider the process

$$dR(s) = (1+\eta_1(\alpha(s)))\lambda_L m_Y ds - (1-a(s))(1+\theta(\alpha(s)))\lambda_L m_Y ds - \lambda_L m_Y a(s) ds + a(s)\sqrt{\lambda_L \sigma_Y} dW_0(s),$$

equivalently, we obtain

$$dR(s) = a(s)\theta(\alpha(s))\lambda_L m_Y ds + a(s)\sqrt{\lambda_L \sigma_Y} dW_0(s). \tag{2.3}$$

2.3 Financial market

Consider an agent facing the problem of portfolio and inter-temporal consumption where the financial market consists of one savings account and N risky securities. The risky securities are stocks and their prices processes $S_1, ..., S_N$ are governed by the following Markov-modulated SDE

$$\begin{cases} dS_n(s) = S_n(s)\left(\mu_n(s,\alpha(s))\,ds + \sum\limits_{m=1}^{N} \sigma_{nm}(s,\alpha(s-))\,dW_m(s)\right), \text{ for } s \in [0,T], \\ S_n(0) > 0, \end{cases} \tag{2.4}$$

where, for $n = 1, 2, ..., N$, $W_n(\cdot)$ is a one dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, the coefficients $\mu_n(\cdot,\cdot) : [0,T] \times \chi \to (0,\infty)$ and $\sigma_n(\cdot,\cdot) = (\sigma_{n1}(\cdot,\cdot), .., \sigma_{nN}(\cdot,\cdot))^\top : [0,T] \times \chi \to \mathbb{R}^N$ represent the appreciation rate and the volatility of the n-th stock, respectively. For brevity, we use $\mu(s,e_i) = (\mu_1(s,e_i), \mu_2(s,e_i), .., \mu_N(s,e_i))^\top$ to indicate the drift rate vector, and the volatility matrix is denoted by $\sigma(s,e_i) = (\sigma_{nm}(s,e_i))_{1\leq n,m\leq N}$.

The price of the savings account is given by the differential equation

$$\begin{cases} dS_0(s) = r_0(s)\,S_0(s)\,ds, \text{ for } s \in [0,T], \\ S_0(0) = 1, \end{cases} \tag{2.5}$$

where $r_0(\cdot)$ is a deterministic function with values in $(0,\infty)$ that represents the interest rate. We suppose that $\mathbb{E}[\mu_n(t,e_i)] > r_0(t) \geq 0$, $dt-a.e.$, for $e_i \in \chi$ and $n = 1, 2, .., N$. This is a very natural supposition, because otherwise, nobody wants to invest in the risky stocks.

2.4 Consumption-reinsurance-investment policies and wealth process

Starting from an initial wealth $x_0 > 0$ and a initial market state $e_{i_0} \in \mathcal{X}$ at time 0. The decision maker is allowed to dynamically invest in the stocks as well as in the bond, consume and purchase proportional reinsurance, throughout the time horizon $[0, T]$. The stochastic process $u(\cdot) = (c(\cdot), a(\cdot), \pi_1(\cdot), \ldots, \pi_N(\cdot))^\top$ describes a trading strategy, where $c(s)$ is the consumption rate at time $s \in [0, T]$, $a(s)$ represents the retention level of reinsurance or new business acquired at time $s \in [0, T]$ and $\pi_n(s)$, for $n = 1, 2, .., N$, is the amount invested in the n-th risky stock at time $s \in [0, T]$. The process $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_N(\cdot))^\top$ represents the investment strategy. The amount invested in the bond at time s is $X^{x_0, e_{i_0}, u}(s) - \sum_{n=1}^{N} \pi_n(s)$, where $X^{x_0, e_{i_0}, u}(\cdot)$ is the wealth process associated with the strategy $u(\cdot)$ and the initial state (x_0, e_{i_0}). The evolution of $X^{x_0, e_{i_0}, u}(\cdot)$ is given by

$$\begin{cases} dX^{x_0, e_{i_0}, u}(s) = dR(s) + \left(X^{x_0, e_{i_0}, u}(s) - \sum_{n=1}^{N} \pi_n(s)\right) \dfrac{dS_0(s)}{S_0(s)} + \sum_{n=1}^{N} \pi_n(s) \dfrac{dS_n(s)}{S_n(s)} \\ \qquad\qquad - c(s)\, ds, \text{ for } s \in [0, T], \\ X^{x_0, u}(0) = x_0, \ \alpha(0) = e_{i_0} \in \chi. \end{cases}$$

Therefore, the wealth process solves the following SDE

$$\begin{cases} dX^{x_0, e_{i_0}, u}(s) = \left\{ r_0(s) X^{x_0, e_{i_0}, u}(s) + \pi(s)^\top r(s, \alpha(s)) + \theta(\alpha(s))\, a(s) \lambda_L m_Y - c(s) \right\} ds \\ \qquad + \sqrt{\lambda_L \sigma_Y}\, a(s)\, dW_0(s) + \pi(s)^\top \sigma(s, \alpha(s))\, dW(s), \text{ for } s \in [0, T], \\ X^{x_0, u}(0) = x_0, \ \alpha(0) = e_{i_0} \in \chi, \end{cases} \tag{2.6}$$

where $W(\cdot) = (W_1(\cdot), ..., W_N(\cdot))$ is a $N-$dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $r(s, e_i) = (\mu_1(s, e_i) - r_0(s), \ldots, \mu_N(s, e_i) - r_0(s))^\top$.

As time evolves, we consider the following controlled stochastic differential equation satisfied by $X(\cdot) = X^{t, \xi, e_i}(\cdot; u(\cdot))$ which parameterized by $(t, \xi, e_i) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \times \chi$

$$\begin{cases} dX(s) = \left\{ r_0(s) X(s) + \pi(s)^\top r(s, \alpha(s)) + \theta(\alpha(s))\, a(s) \lambda_L m_Y - c(s) \right\} ds + \sqrt{\lambda_L \sigma_Y}\, a(s)\, dW_0(s) \\ \qquad + \pi(s)^\top \sigma(s, \alpha(s))\, dW(s), \text{ for } s \in [t, T], \\ X(t) = \xi, \ \alpha(t) = e_i. \end{cases} \tag{2.7}$$

Definition 1 (Admissible Strategy) *A strategy $u(\cdot) = \left(c(\cdot), a(\cdot), \pi(\cdot)^\top\right)^\top$ is said to be admissible over $[t, T]$ if $u(\cdot) \in \mathcal{M}^1_{\mathcal{F}}(t, T; \mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}\left(t, T; \mathbb{R}^{N+1}\right)$ and for any $(t, \xi, e_i) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \times \chi$, the equation (2.7) has a unique solution $X(\cdot) = X^{t, \xi, e_i}(\cdot; u(\cdot)) \geq 0$.*

Remark 2 *Any component of the vector $\pi(\cdot)$ may become negative, which is to be interpreted as short-selling that particular stock. The amount $X(s) - \sum_{n=1}^{N} \pi_n(s)$ invested in the bond at time s may also become negative, and this corresponds to borrowing at the interest rate $r(s, e_i)$, for $i = 1, ..., N$, see Remark 2.3 of Karatzas et al. [27].*

We make the following assumption about the coefficients,

(H1) The maps $r_0(\cdot), r(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are uniformly bounded, we also assume the uniform ellipticity condition as follow:

$$\sigma(s, e_i)\, \sigma(s, e_i)^\top \geq \epsilon I_N, \ \forall (s, e_i) \in [0, T] \times \chi,$$

for some $\epsilon > 0$, where I_N denotes the identity matrix on $\mathbb{R}^{N \times N}$.

Under the bondedness condition on the coefficients in **(H1)**, for any $(t, \xi, e_i) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \times \chi$ and $u(\cdot) \in \mathcal{M}^1_{\mathcal{F}}(t, T; \mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}\left(t, T; \mathbb{R}^{N+1}\right)$, the controlled state equation (2.7) admit a unique solution $X(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(t, T; \mathbb{R})$. Furthermore, we have the estimate

$$\mathbb{E}\left[\sup_{t \leq s \leq T} |X(s)|^2\right] \leq K\left(1 + \mathbb{E}\left[|\xi|^2\right]\right), \tag{2.8}$$

for some positive constant K. In particular for $t=0$, $x_0>0$ and $u(\cdot)=\left(c(\cdot),a(\cdot),\pi(\cdot)^{\top}\right)^{\top}\in\mathcal{M}_{\mathcal{F}}^{1}(0,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)$, the controlled state equation (2.6) admit a unique solution $X^{x_0,e_{i_0},u}(\cdot)\in\mathcal{L}_{\mathcal{F}}^{2}(0,T;\mathbb{R})$ also we have the following estimate

$$\mathbb{E}\left[\sup_{0\leq s\leq T}\left|X^{x_0,e_{i_0},u}(s)\right|^2\right]\leq K\left(1+|x_0|^2\right). \tag{2.9}$$

2.5 General discounted utility function

It is worth to noting that, the rate of time preference is considered to be constant in most financial works (This means that the discount is exponential). Nevertheless, there is mounting evidence that this may not be true. In this section, we discuss general discounting preferences. We also present the essential modeling framework for the Merton consumption and portfolio problem under regime-switching for surplus-dependent reinsurance. For additional information about the classic Merton model, we refer the reader to [20], [27], [36] and [37].

2.5.1 Discount function

Most works use a specific form of the non-exponential discount factor when discounting is non-exponential. In contrast to these studies, we consider the discount function from the general form

Definition 3 *A discount function $\Im(\cdot)$ is a continuous, deterministic function satisfying $\Im(0)=1$, $\Im(s)>0$ $ds-a.e.$ and $\int_0^T\Im(s)\,ds<\infty$.*

Many articles provide examples of discount functions, including exponential discount functions, see [36] and [37], mixtures of exponential functions, see [17], and hyperbolic discount functions, see [54].

2.5.2 Utility functions and objective

The insurer extracts utility from inter-temporal consumption and terminal wealth in order to assess the performance of a trading strategy. The utility of inter-temporal consumption is represented by $\vartheta(\cdot)$ and the utility of the terminal wealth at some non-random horizon T is represented by $h(\cdot)$. Then, for any $(t,\xi,e_i)\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})\times\chi$ the consumption-reinsurance-investment optimization problem is denoted as the following: maximize

$$J(t,\xi,e_i;u(\cdot))=\mathbb{E}^t\left[\int_t^T\Im(s-t)\,\vartheta(c(s))\,ds+\Im(T-t)\,h(X(T))\right], \tag{2.10}$$

over $u(\cdot)\in\mathcal{M}_{\mathcal{F}}^{1}(t,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)$, subject to (2.7), where $\mathbb{E}^t[\cdot]=\mathbb{E}[\cdot\,|\mathcal{F}_t]$.

We impose the following conditions on the utility functions.

(H2) The maps $\vartheta(\cdot),h(\cdot):\mathbb{R}\to\mathbb{R}$ are strictly concave, strictly increasing and satisfy the integrability condition

$$\mathbb{E}\left[\int_0^T|\vartheta(c(s))|\,ds+|h(X(T))|\right]<\infty.$$

(H3) The maps $\vartheta(\cdot),h(\cdot)$ are twice continuously differentiable, in addition, the derivatives $\vartheta_x(\cdot),h_x(\cdot),\vartheta_{xx}(\cdot)$ and $h_{xx}(\cdot)$ are continuous.

(H4) For all admissible strategy pairs, there exists a constant $p>1$ sush that

$$\mathbb{E}\left[\int_0^T|\vartheta_x(c(s))|^p\,ds+|h_x(X(T))|^p\right]<\infty,$$

$$\mathbb{E}\left[\int_0^T\sup_{\eta\in\mathbb{R},|\eta|\leq M}|\vartheta_{xx}(c(s)+\eta)|^p\,ds\right]<\infty,\ \text{for } M\geq 0.$$

In the rest of the paper, we write $W^{\sharp}(s)=\left(0,W^{\star}(s)^{\top}\right)^{\top}$ where $W^{\star}(s)=\left(W_0(s),W(s)^{\top}\right)^{\top}$. We denote $B(s,\alpha(s))=\left(-1,\theta(\alpha(s))\lambda_L m_Y,r(s,\alpha(s))^{\top}\right)^{\top}$ and $\pounds=\left(1,0_{\mathbb{R}^{N+1}}^{\top}\right)^{\top}$, we also consider the following matrix notation

$$\widetilde{\sigma}(s,\alpha(s))=\begin{pmatrix}\sqrt{\lambda_L\sigma_Y} & 0_{\mathbb{R}^N}^{\top}\\ 0_{\mathbb{R}^N} & \sigma(s,\alpha(s))\end{pmatrix} \text{ and } D(s,\alpha(s))=\begin{pmatrix}0 & 0_{\mathbb{R}^{N+1}}^{\top}\\ 0_{\mathbb{R}^{N+1}} & \widetilde{\sigma}(s,\alpha(s))\end{pmatrix},$$

then the optimal control problem associated with (2.7) and (2.10) is equivalent to maximize

$$J(t,\xi,e_i;u(\cdot))=\mathbb{E}^t\left[\int_t^T \Im(s-t)\,\vartheta\left(\pounds^{\top}u(\cdot)\right)ds+\Im(T-t)\,h(X(T))\right], \tag{2.11}$$

subject to

$$\begin{cases} dX(s)=\left\{r_0(s)X(s)+u(s)^{\top}B(s,\alpha(s))\right\}ds+u(s)^{\top}D(s,\alpha(s))\,dW^{\sharp}(s), \text{ for } s\in[t,T],\\ X(t)=\xi,\ \alpha(t)=e_i,\end{cases} \tag{2.12}$$

over $u(\cdot)\in\mathcal{M}_{\mathcal{F}}^1(t,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^2\left(t,T;\mathbb{R}^{N+1}\right)$.

3 Equilibrium strategies

It is clear that the problem described by (2.11) − (2.12) is time-inconsistent in the sense that the Bellman optimality principle does not hold, since a restriction of the optimal strategy for a given starting pair at a subsequent time interval might not be optimal for that corresponding starting pair. See Ekeland & Pirvu [17] and Yong [52] for a more detailed explanation. We consider open-loop Nash equilibrium controls rather than optimal controls due to the lack of time consistency. These are strategies which are optimal to implement now given that they will be implemented in the future. Suppose that every player s, such that $s>t$, will use the strategy $\hat{u}(s)$. Then the optimal choice for player t is that, he/she also uses the strategy $\hat{u}(t)$.

Nevertheless, the problem with this "definition", is that the individual player t does not really influence the outcome of the game at all. He/she only chooses the control at the single point t, and since this is a time set of Lebesgue measure zero, the control dynamics will not be influenced. Similarly to [25], we define an equilibrium by local spike variation, given for $t\in[0,T]$, an admissible trading strategy $\hat{u}(\cdot)\in\mathcal{M}_{\mathcal{F}}^1(t,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^2\left(t,T;\mathbb{R}^{N+1}\right)$. For any $\mathbb{R}^{N+2}-$valued, $\mathcal{F}_t-$measurable and bounded random variable v and for any $\varepsilon>0$, consider

$$u^{\varepsilon}(s):=\begin{cases}\hat{u}(s)+v, & \text{for } s\in[t,t+\varepsilon),\\ \hat{u}(s), & \text{for } s\in[t+\varepsilon,T].\end{cases} \tag{3.1}$$

We have the following definition.

Definition 4 (Open-loop Nash equilibrium) *An admissible strategy $\hat{u}(\cdot)\in\mathcal{M}_{\mathcal{F}}^1(t,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^2\left(t,T;\mathbb{R}^{N+1}\right)$ is called an open-loop Nash equilibrium strategy if for every sequence $\varepsilon_n\downarrow 0$, we have*

$$\lim_{\varepsilon_n\downarrow 0}\frac{1}{\varepsilon_n}\left\{J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon_n}(\cdot)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right)\right\}\leq 0, \tag{3.2}$$

for any $t\in[0,T]$, where $\hat{X}(\cdot)$ is the equilibrium wealth process solution of the SDE

$$\begin{cases} d\hat{X}(s)=\left\{r_0(s)\hat{X}(s)+\hat{u}(s)^{\top}B(s,\alpha(s))\right\}ds+\hat{u}(s)^{\top}D(s,\alpha(s))\,dW^{\sharp}(s), \text{ for } s\in[t,T],\\ \hat{X}(t)=\xi,\alpha(t)=e_i.\end{cases} \tag{3.3}$$

3.1 A necessary and sufficient condition for equilibrium controls

In this work, we follow an alternative approach, which is basically the constriction of necessary and sufficient condition for equilibrium. In the same spirit as demonstrating the stochastic Pontryagin's maximum principle for equilibrium in [25] for linear quadratic models case. This condition is derived by a second-order expansion in the spike variation.

We shall now present the adjoint equations, that are used to characterize the open-loop Nash equilibrium controls.

3.1.1 Adjoint processes

Let $\hat{u}(\cdot)=\left(\hat{c}(\cdot),\hat{a}(\cdot),\hat{\pi}(\cdot)^{\top}\right)^{\top}\in\mathcal{M}_{\mathcal{F}}^{1}(0,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)$ an admissible strategy and denote by $\hat{X}(\cdot)\in\mathcal{L}_{\mathcal{F}}^{2}(0,T;\mathbb{R})$ the corresponding wealth process. For each $t\in[0,T]$, the first order adjoint equation satisfied by the processes $(p(\cdot;t),q(\cdot;t),l(\cdot;t))$ and defined on the time interval $[t,T]$ by the following linear backward SDE

$$\begin{cases} dp(s;t)=-r_0(s)\,p(s;t)\,ds+\sum\limits_{m=1}^{N+1}q_m(s;t)\,dW_m(s)+\sum\limits_{j\neq i}l_{ij}(s;t)\,dJ^{ij}(s)\,,\ s\in[t,T]\,,\\ p(T;t)=\Im(T-t)\,h_x\left(\hat{X}(T)\right), \end{cases} \tag{3.4}$$

where $q(\cdot;t)=(q_0(\cdot;t),q_1(\cdot;t),\ldots,q_N(\cdot;t))^{\top}$ and $l(s;t)=(l_{ij}(s;t))_{1\leq i,j\leq D}\in\mathbb{R}^{D\times D}$. According to Theorem 5.15 in [31], for any $\hat{u}(\cdot)\in\mathcal{M}_{\mathcal{F}}^{1}(0,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)$ and $t\in[0,T]$, we deduce that equation (3.4) has a unique adapted solution $(p(\cdot;\cdot),q(\cdot;\cdot),l(\cdot;\cdot))\in\left(\mathcal{L}_{\mathcal{F}}^{2}(t,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)\times\mathcal{M}_{\mathcal{F},p}^{g,2}\left(t,T;\mathbb{R}^{D\times D}\right)\right)$. Moreover there exists a constant $K>0$ such that, for any $t\in[0,T]$, we obtain the following estimate

$$\|p(\cdot;t)\|^{2}_{\mathcal{L}_{\mathcal{F}}^{2}(t,T;\mathbb{R})}+\|q(\cdot;t)\|^{2}_{\mathcal{M}_{\mathcal{F}}^{2}(t,T;\mathbb{R}^{N+1})}+\|l(\cdot;t)\|^{2}_{\mathcal{M}_{\mathcal{F},p}^{g,2}(t,T;\mathbb{R}^{D\times D})}\leq K\left(1+\xi^{2}\right). \tag{3.5}$$

The second order adjoint equation satisfied by the processes $(P(\cdot;t),Q(\cdot;t),L(\cdot;t))$ and defined on the time interval $[t,T]$ by the following linear backward SDE

$$\begin{cases} dP(s;t)=-2r_0(s)\,P(s;t)\,ds+\sum\limits_{m=1}^{N+1}Q_m(s;t)\,dW_m(s)+\sum\limits_{j\neq i}L_{ij}(s;t)\,dJ^{ij}(s)\,,\ s\in[t,T]\,,\\ P(T;t)=\Im(T-t)\,h_{xx}\left(\hat{X}(T)\right), \end{cases} \tag{3.6}$$

where $Q(\cdot;t)=(Q_0(\cdot;t),Q_1(\cdot;t),\ldots,Q_N(\cdot;t))^{\top}$ and $L(s;t)=(L_{ij}(s;t))_{1\leq i,j\leq D}\in\mathbb{R}^{D\times D}$. According to Theorem 5.15 in [31], the above BSDE admits a unique solution

$$(P(\cdot;t),Q(\cdot;t),L(\cdot;t))\in\left(\mathcal{L}_{\mathcal{F}}^{2}(t,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)\times\mathcal{M}_{\mathcal{F},p}^{g,2}\left(t,T;\mathbb{R}^{D\times D}\right)\right).$$

In addition, $P(\cdot;t)$ has the following representation.

$$P(s;t)=\mathbb{E}^{s}\left[\Im(T-t)\,e^{\int_s^T 2r_0(\tau)d\tau}h_{xx}\left(\hat{X}(T)\right)\right],\ \text{for } s\in[t,T]. \tag{3.7}$$

In fact, if we define the function $\Theta(\cdot,t)$, for each $t\in[0,T]$, as the solution of the following linear ODE

$$\begin{cases} d\Theta(\tau,t)=r_0(\tau)\,\Theta(\tau,t)\,d\tau,\ \text{for } \tau\in[t,T]\,,\\ \Theta(t,t)=1, \end{cases} \tag{3.8}$$

and we apply the Itô's formula to $\tau\to P(\tau;t)\,\Theta(\tau,t)^{2}$ on $[t,T]$, by taking conditional expectations, we get (3.7). It's worth mentioning that, since $h_{xx}\left(\hat{X}(T)\right)\leq 0$, then $P(s;t)\leq 0$, $ds-a.e.$

3.1.2 A characterization of equilibrium strategies

The following theorem presents the first main result of this work, it gives a necessary and sufficient condition for equilibrium. First, we define the process $\tilde{q}(s;t)=\left(0,q(s;t)^{\top}\right)^{\top}$ and we adopt the following notations

$$\mathcal{H}(s;t)\triangleq p(s;t)\,B(s,\alpha(s))+D(s,\alpha(s))\,\tilde{q}(s;t)+\Im(s-t)\,\vartheta_x\left(\pounds^{\top}\hat{u}(s)\right)\pounds, \tag{3.9}$$

and

$$\mathcal{A}^{\varepsilon}(s;t)\triangleq\begin{pmatrix}\Im(s-t)\,\vartheta_{xx}\left(\pounds^{\top}\left(\hat{u}(s)+\lambda v1_{[t,t+\varepsilon)}\right)\right) & 0_{\mathbb{R}^{N+1}}^{\top}\\ 0_{\mathbb{R}^{N+1}} & \widetilde{\sigma}(s,\alpha(s))\,\widetilde{\sigma}(s,\alpha(s))^{\top}P(s;t)\end{pmatrix}. \tag{3.10}$$

Theorem 5 *Let **(H1)-(H4)** hold. Given an admissible strategy* $\hat{u}(\cdot) \in \mathcal{M}^1_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$, *let for any* $t \in [0,T]$, *the process*

$$\left(p(\cdot;t), q(\cdot;t), l(\cdot;t)\right) \in \left(\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right) \times \mathcal{M}^{g,2}_{\mathcal{F},p}\left(t,T;\mathbb{R}^{D\times D}\right)\right)$$

be the unique solution to the BSDE (3.4). *Then,* $\hat{u}(\cdot)$ *is an equilibrium trading strategy, if and only if, the following condition holds*

$$\mathcal{H}(t;t) = 0,\ d\mathbb{P}\text{-}a.s.,\ dt - a.e. \tag{3.11}$$

To prove this theorem, we need to derive some technical results. Denote by $\hat{X}^\varepsilon(\cdot)$ the solution of the state equation corresponding to $u^\varepsilon(\cdot)$. It follows from the standard perturbation approach see e.g. [53] that

$$\hat{X}^\varepsilon(s) - \hat{X}(s) = y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s),\ \text{for } s \in [t,T], \tag{3.12}$$

where for any $\mathbb{R}^{N+2}$−valued, $\mathcal{F}_t$−measurable and bounded random variable v and for any $\varepsilon \in [0, T-t)$, $y^{\varepsilon,v}(\cdot)$ and $z^{\varepsilon,v}(\cdot)$ solve the following linear SDEs, respectively

$$\begin{cases} dy^{\varepsilon,v}(s) = r_0(s)\, y^{\varepsilon,v}(s)\, ds + v^\top D(s,\alpha(s))\, 1_{[t,t+\varepsilon)}(s)\, dW^\sharp(s),\ \text{for } s \in [t,T], \\ y^{\varepsilon,v}(t) = 0, \end{cases} \tag{3.13}$$

and

$$\begin{cases} dz^{\varepsilon,v}(s) = \left\{r_0(s)\, z^{\varepsilon,v}(s) + v^\top B(s,\alpha(s))\, 1_{[t,t+\varepsilon)}(s)\right\} ds,\ \text{for } s \in [t,T], \\ z^{\varepsilon,v}(t) = 0. \end{cases} \tag{3.14}$$

Proposition 6 *Let **(H1)-(H4)** holds. The following estimates hold for any* $k \geq 1$ *and* $t \in [0,T]$

$$\mathbb{E}^t\left[\sup_{s\in[t,T]} \left|y^{\varepsilon,v}(s)\right|^{2k}\right] = O\left(\varepsilon^k\right), \tag{3.15}$$

$$\mathbb{E}^t\left[\sup_{s\in[t,T]} \left|z^{\varepsilon,v}(s)\right|^{2k}\right] = O\left(\varepsilon^{2k}\right), \tag{3.16}$$

$$\mathbb{E}^t\left[\sup_{s\in[t,T]} \left|y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s)\right|^{2k}\right] = O\left(\varepsilon^k\right). \tag{3.17}$$

Furthermore, we have the following equality

$$\begin{aligned} &J\left(t, \hat{X}(t), \alpha(t); u^\varepsilon(\cdot)\right) - J\left(t, \hat{X}(t), \alpha(t); \hat{u}(\cdot)\right) \\ &= \int_t^{t+\varepsilon} \mathbb{E}^t\left[\langle \mathcal{H}(s;t), v\rangle + \frac{1}{2}\langle \mathcal{A}^\varepsilon(s;t)\, v, v\rangle\right] ds + o(\varepsilon). \end{aligned} \tag{3.18}$$

Proof. See the Appendix. ■

We now introduce the following technical lemma , which we will need later. The proof is based on an argument inspired by Hamaguchi [23].

Lemma 7 *Under assumptions **(H1)-(H4)**, there exists a sequence* $(\varepsilon^t_n)_{n\in\mathbb{N}} \subset (0, T-t)$ *satisfying* $\varepsilon^t_n \to 0$ *as* $n \to \infty$, *such that*

1) $\displaystyle\lim_{n\to\infty} \frac{1}{\varepsilon^t_n} \int_t^{t+\varepsilon^t_n} \mathbb{E}^t\left[\mathcal{H}(s;t)\right] ds = \mathcal{H}(t;t),\ d\mathbb{P} - a.s,\ dt - a.e.$

2) $\displaystyle\lim_{n\to\infty} \frac{1}{\varepsilon^t_n} \int_t^{t+\varepsilon^t_n} \mathbb{E}^t\left[\mathcal{A}^{\varepsilon^t_n}(s;t)\right] ds = \mathcal{A}^0(t;t),\ d\mathbb{P} - a.s,\ dt - a.e.$

Proof. See the Appendix. ■

Proof of Theorem 5. Given an admissible strategy $\hat{u}(\cdot) \in \mathcal{M}_{\mathcal{F}}^{1}(0,T;\mathbb{R}) \times \mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)$, for which (3.11) holds, from Lemma 7, we have according to (3.18), for any $t \in [0,T]$ and for any $\mathbb{R}^{N+2}-$valued, $\mathcal{F}_t-$measurable and bounded random variable v, there exists a sequence $\left(\varepsilon_n^t\right)_{n\in\mathbb{N}} \subset (0,T-t)$ satisfying $\varepsilon_n^t \to 0$ as $n \to \infty$, such that

$$\begin{aligned}\lim_{n\to 0}\frac{1}{\varepsilon_n^t}\left\{J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon}(\cdot)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right)\right\} &= \langle\mathcal{H}(t;t),v\rangle+\frac{1}{2}\left\langle\mathcal{A}^0(t;t)v,v\right\rangle,\\ &= \frac{1}{2}\left\langle\mathcal{A}^0(t;t)v,v\right\rangle,\\ &\leq 0,\end{aligned}$$

where the last inequality is due to the concavity condition of $\vartheta(\cdot)$ and $h(\cdot)$. Then $\hat{u}(\cdot)$ is an equilibrium strategy.

Conversely, suppose that $\hat{u}(\cdot)$ is an equilibrium strategy. Hence, from (3.2) together with (3.18) and Lemma 7, for any $(t,u) \in [0,T] \times \mathbb{R}^{N+2}$, we get

$$\langle\mathcal{H}(t;t),u\rangle+\frac{1}{2}\left\langle\mathcal{A}^0(t;t)u,u\right\rangle \leq 0. \tag{3.19}$$

Now, we define $\forall (t,u) \in [0,T]\times\mathbb{R}^{N+2}$, $\Xi(t,u) = \langle\mathcal{H}(t;t),u\rangle + \frac{1}{2}\left\langle\mathcal{A}^0(t;t)u,u\right\rangle$. Easy manipulations demonstrating that the inequality (3.19) is equivalent to

$$\Xi(t,0) = \max_{u\in\mathbb{R}^{N+2}}\Xi(t,u),\ d\mathbb{P}-a.s,\forall t\in[0,T]. \tag{3.20}$$

Thus, the following condition results from the maximum condition (3.20)

$$\Xi_u(t,0) = \mathcal{H}(t;t) = 0,\ d\mathbb{P}-a.s,\forall t\in[0,T]. \tag{3.21}$$

This completes the proof.

3.2 A characterization of equilibrium strategies by verification argument

The sufficient condition of optimality plays an important role for computing optimal controls in classical stochastic control theory (time-consistent). It asserts that if an admissible control fulfills the maximum condition of the Hamiltonian, then it is in fact optimal for the stochastic control problem. This allows solving examples of optimal control problems where a smooth solution to the associated adjoint equation can be found.

The objective of the following theorem is to characterize the open-loop equilibrium strategies just by a sufficient equilibrium condition. First, in order to overcome the technical difficulties mentioned by the hypothesis (H3) in the practice, let us consider the following condition about the utility functions,

(H3') The maps $\vartheta(\cdot), h(\cdot)$ are continuously differentiable and the first order derivatives $\vartheta_x(\cdot), h_x(\cdot)$ are continuous.

We have the following theorem

Theorem 8 *Let **(H1), (H2)** and **(H3')** hold. Given an admissible strategy $\hat{u}(\cdot) \in \mathcal{M}_{\mathcal{F}}^{1}(0,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)$, let for any $t\in[0,T]$, the process*

$$(p(\cdot;t),q(\cdot;t),l(\cdot;t)) \in \left(\mathcal{L}_{\mathcal{F}}^{2}(t,T;\mathbb{R})\times\mathcal{M}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{N+1}\right)\times\mathcal{M}_{\mathcal{F},p}^{g,2}\left(t,T;\mathbb{R}^{D\times D}\right)\right),$$

be the unique solution to the BSDE (3.4)*. Then, $\hat{u}(\cdot)$ is an equilibrium trading strategy, if the following condition holds*

$$\mathcal{H}(t;t) = 0,\ d\mathbb{P}-a.s.,\ dt-a.e. \tag{3.22}$$

Proof. Assume that $\hat{u}(\cdot)$ is an admissible control for which the condition (3.22) holds. Furthermore, for any $t \in [0,T]$ and $\varepsilon \in [0, T-t)$, we consider $u^{\varepsilon}(\cdot)$ by (3.1). Then, we have the following difference

$$J\left(t, \hat{X}(t), \alpha(t); \hat{u}(\cdot)\right) - J\left(t, \hat{X}(t), \alpha(t); u^{\varepsilon}(\cdot)\right)$$
$$= \mathbb{E}^t\left[\int_t^T \Im(s-t)\left(\vartheta\left(\pounds^{\top}\hat{u}(s)\right) - \vartheta\left(\pounds^{\top}u^{\varepsilon}(s)\right)\right)ds + \Im(T-t)\left(h\left(\hat{X}(T)\right) - h\left(\hat{X}^{\varepsilon}(T)\right)\right)\right].$$

Mentioning that the concavity of $h(\cdot)$ gives us

$$\mathbb{E}^t\left[\Im(T-t)\left(h\left(\hat{X}(T)\right) - h\left(\hat{X}^{\varepsilon}(T)\right)\right)\right] \geq \mathbb{E}^t\left[\Im(T-t)\left(\hat{X}(T) - \hat{X}^{\varepsilon}(T)\right)^T h_x\left(\hat{X}(T)\right)\right].$$

Consequently, by the terminal condition in the BSDE (3.4) we get that

$$J\left(t, \hat{X}(t), \alpha(t); \hat{u}(\cdot)\right) - J\left(t, \hat{X}(t), \alpha(t); u^{\varepsilon}(\cdot)\right)$$
$$\geq \mathbb{E}^t\left[\int_t^T \Im(s-t)\left(\vartheta\left(\pounds^{\top}\hat{u}(s)\right) - \vartheta\left(\pounds^{\top}u^{\varepsilon}(s)\right)\right)ds + \left(\hat{X}(T) - \hat{X}^{\varepsilon}(T)\right)^T p(T;t)\right]. \tag{3.23}$$

By applying Ito's formula to $s \mapsto \left(\hat{X}(s) - \hat{X}^{\varepsilon}(s)\right)^T p(s;t)$ on $[t,T]$, we obtain

$$\mathbb{E}^t\left[\left(\hat{X}(T) - \hat{X}^{\varepsilon}(T)\right)^T p(T;t)\right]$$
$$= \mathbb{E}^t\left[\int_t^T \left(\hat{u}(s) - u^{\varepsilon}(s)\right)^T \left(B(s,\alpha(s))\,p(s;t) + D(s,\alpha(s))\,\widetilde{q}(s;t)\right)ds\right]. \tag{3.24}$$

By the concavity condition of $\vartheta(\cdot)$, we find that

$$\mathbb{E}^t\left[\int_t^T \Im(s-t)\left(\vartheta\left(\pounds^{\top}\hat{u}(s)\right) - \vartheta\left(\pounds^{\top}u^{\varepsilon}(s)\right)\right)ds\right] \geq \mathbb{E}^t\left[\int_t^T \Im(s-t)\left\langle\vartheta_x\left(\pounds^{\top}\hat{u}(s)\right)\pounds, \hat{u}(s) - u^{\varepsilon}(s)\right\rangle ds\right]. \tag{3.25}$$

By taking (3.24) and (3.25) in (3.23), it follows that

$$J\left(t, \hat{X}(t), \alpha(t); u^{\varepsilon}(\cdot)\right) - J\left(t, \hat{X}(t), \alpha(t); \hat{u}(\cdot)\right)$$
$$\leq \mathbb{E}^t\left[\int_t^T \left\langle B(s,\alpha(s))\,p(s;t) + D(s,\alpha(s))\,\widetilde{q}(s;t) + \Im(s-t)\,\vartheta_x\left(\pounds^{\top}\hat{u}(s)\right)\pounds, u^{\varepsilon}(s) - \hat{u}(s)\right\rangle\right]ds.$$

Using (3.9) and dividing both sides by ε then taking the limit when ε vanishes, we conclude by Lemma 7 that $\hat{u}(\cdot)$ is an equilibrium control. ■

Remark 9 *The goal of the sufficient condition of optimality is to find an optimal control by calculating the difference* $J(\hat{u}(\cdot)) - J(u(\cdot))$ *in terms of the Hamiltonian function, where* $u(\cdot)$ *is an arbitrary admissible control. Here, the spike variation perturbation* (3.1) *plays a major role in deriving the sufficient condition for equilibrium strategies, which reduces to the calculation of the difference* $J\left(t, \hat{X}(t), \alpha(t), \hat{u}(\cdot)\right) - J\left(t, \hat{X}(t), \alpha(t), u^{\varepsilon}(\cdot)\right)$, *without the need to achieving the second order expansion in the spike variation.*

4 Equilibrium strategies and related partial differential-difference equation

From theorems 5 and 8, we conclude that we can get equilibrium trading strategy by solving a system of FBSDEs which is not standard since the "flow" of the unknown process $\left(p(\cdot;t), q(\cdot;t), l(\cdot;t)\right)_{t\in[0,T]}$ is involved.

Furthermore, there is an additional constraint acting on the "diagonal" (i.e. when $s = t$) of the flow. To the best of our knowledge, the explicit ability to solve this type of equation remains an open problem, except for a certain small-time solvability results see [24]. For an open-loop equilibrium strategy, Hamaguchi in [23] presented some equilibrium conditions for general a time-inconsistent investment and consumption model in a possibly incomplete market under general discount functions with random endowments. These conditions are connected to the solvability of an equivalent fully coupled FBSDE system, which is more feasible than a flow of FBSDEs.

In this section, we define the equilibrium rule, and then we derive a parabolic backward PDDE. Our PDDE is comparable with the one obtained in Pirvu & Zhang [42], for some particular discount functions in a finite horizon with different utility functions.

In this section, let us look at the regime switching Merton's portfolio problem with general discounting. First, we consider the following parabolic backward partial differential equation

$$\begin{cases} -\text{ß}_t(t,x,e_i) = \text{ß}_x(t,x,e_i)\left(r_0(t)\,x - \tilde{B}(t,e_i)^\top \Sigma(t,e_i)\,\tilde{B}(t,e_i)\,\dfrac{\text{ß}(t,x,e_i)}{\text{ß}_x(t,x,e_i)} - \mathcal{I}\left(\Im(T-t)\,\text{ß}(t,x,e_i)\right)\right) \\ \qquad\qquad + \dfrac{1}{2}\text{ß}_{xx}(t,x,e_i)\,\tilde{B}(t,e_i)^\top \Sigma(t,e_i)\,\tilde{B}(t,e_i)\left(\dfrac{\text{ß}(t,x,e_i)}{\text{ß}_x(t,x,e_i)}\right)^2 + r_0(t)\,\text{ß}(t,x,e_i) \\ \qquad\qquad + \sum\limits_{j=1}^{D} g_{ij}\left[\text{ß}(t,x,e_j) - \text{ß}(t,x,e_i)\right], \quad (t,x,e_i) \in [0,T]\times\mathbb{R}\times\chi, \\ \text{ß}(T,x,e_i) = h_x(x), \end{cases} \tag{4.1}$$

where we denote by $\mathcal{I}(\cdot)$ the inverse function of the strictly decreasing marginal derivative utility $\vartheta_x(\cdot)$, $\tilde{B}(s,\alpha(s)) = \left(\theta(\alpha(s))\,\lambda_L m_Y, r(s,\alpha(s))^\top\right)^\top$ and $\Sigma(s,\alpha(s)) \equiv \left(\widetilde{\sigma}(s,\alpha(s))\,\widetilde{\sigma}(s,\alpha(s))^\top\right)^{-1}$.

Now we will introduce the verification theorem

Theorem 10 *Let **(H1)-(H4)** hold. If there exists a classical solution*

$$\text{ß}(\cdot,\cdot,e_i) \in \mathcal{C}^{1,2}((0,T)\times\mathbb{R},\mathbb{R}) \cap \mathcal{C}([0,T]\times\mathbb{R},\mathbb{R}) \ \textit{for each } e_i \in \chi$$

of the PDE (4.1) *such that the stochastic differential equation*

$$\begin{cases} d\hat{X}(s) = & \left\{ r_0(s)\,\hat{X}(s) - \tilde{B}(s,\alpha(s))^\top \Sigma(s,\alpha(s))\,\tilde{B}(s,\alpha(s))\,\dfrac{\text{ß}\left(s,\hat{X}(s),\alpha(s)\right)}{\text{ß}_x\left(s,\hat{X}(s),\alpha(s)\right)} \right. \\ & \left. -\mathcal{I}\left(\Im(T-s)\,\text{ß}\left(s,\hat{X}(s),\alpha(s)\right)\right)\right\} ds \\ & -\dfrac{\text{ß}\left(s,\hat{X}(s),\alpha(s)\right)}{\text{ß}_x\left(s,\hat{X}(s),\alpha(s)\right)}\tilde{B}(s,\alpha(s))^\top \Sigma(s,\alpha(s))\,\widetilde{\sigma}(s,\alpha(s))\,dW^\star(s), \ \ s\in[0,T], \\ \hat{X}(0) = & x_0, \alpha(0) = e_{i_0} \in \chi, \end{cases} \tag{4.2}$$

has a unique solution $\hat{X}(\cdot)$, where the following estimate holds

$$\mathbb{E}\left[\sup_{0\le t\le T}|X(t)|^2\right] \le K\left(1+|x_0|^2\right),$$

then, the equilibrium trading strategy $\hat{u}(\cdot) = \left(\hat{c}(\cdot),\hat{a}(\cdot),\hat{\pi}(\cdot)^\top\right)^\top$ is given by

$$\hat{c}(t) = \mathcal{I}\left(\Im(T-t)\,\text{ß}\left(t,\hat{X}(t),\alpha(t)\right)\right), \ dt - a.e., \tag{4.3}$$

$$\hat{a}(t) = -\frac{\theta(\alpha(t))\,m_Y\,\text{ß}\left(t,\hat{X}(t),\alpha(t)\right)}{\sigma_Y\,\text{ß}_x\left(t,\hat{X}(t),\alpha(t)\right)}, \ dt - a.e., \tag{4.4}$$

$$\hat{\pi}(t) = -\frac{r(t,\alpha(t))\,\text{ß}\left(t,\hat{X}(t),\alpha(t)\right)}{\sigma(t,\alpha(t))\,\sigma(t,\alpha(t))^\top\,\text{ß}_x\left(t,\hat{X}(t),\alpha(t)\right)}, \ dt - a.e. \tag{4.5}$$

Proof. Suppose that $\hat{u}(\cdot) = \left(\hat{c}(\cdot), \hat{M}(\cdot)^\top\right)^\top$ is an equilibrium control, where $\hat{M}(\cdot) = \left(\hat{a}(\cdot), \hat{\pi}(\cdot)^\top\right)^\top$ and denote by $\hat{X}(\cdot)$ the corresponding wealth process. Then, in view of Theorem 8, there exist an adapted process $\left(\hat{X}(\cdot), (p(\cdot;t), q(\cdot;t), l(\cdot;t))_{t\in[0,T]}\right)$ that satisfies the following system of regime switching forward-backward stochastic differential equations,

$$\begin{cases} d\hat{X}(s) = \left\{r_0(s)\hat{X}(s) + \hat{M}(s)^\top \tilde{B}(s,\alpha(s)) - \hat{c}(s)\right\} ds + \hat{M}(s)^\top \widetilde{\sigma}(s,\alpha(s))\, dW^\star(s), s \in [t,T], \\ dp(s;t) = -r_0(s)\, p(s;t)\, ds + q(s;t)^\top dW^\star(s) + \sum\limits_{j\neq i} l_{ij}(s;t)\, dJ^{ij}(s), 0 \leq t \leq s \leq T, \\ \hat{X}(0) = x_0, \ , \alpha(0) = e_{i_0} \in \chi, \\ p(T;t) = \Im(T-t)\, h_x\left(\hat{X}(T)\right), \ t \in [0,T], \end{cases} \tag{4.6}$$

with conditions

$$p(t;t) - \vartheta_x(\hat{c}(t)) = 0, \ dt - a.e., \tag{4.7}$$

$$p(t;t)\, \tilde{B}(t,\alpha(t)) + \widetilde{\sigma}(t,\alpha(t))\, q(t;t) = 0, \ dt - a.e. \tag{4.8}$$

Now, we consider the following ansatz from the terminal condition in the first order adjoint process

$$p(s;t) = \Im(T-t)\, \mathcal{V}\left(s, \hat{X}(s), \alpha(s)\right), \forall\ 0 \leq t \leq s \leq T, \tag{4.9}$$

for some deterministic function $\mathcal{V}(\cdot,\cdot,e_i) \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$, for each $e_i \in \chi$ such that $\mathcal{V}(T,\cdot,e_i) = h_x(\cdot)$.

We apply the integration by parts formula (see, e.g., [15]) to (4.9), which yields

$$\begin{aligned} & dp(s;t) \\ = \ & \sum_{i=1}^{D} \langle \alpha(s-), e_i \rangle \Bigg(\Im(T-t) \Bigg\{ \mathcal{V}_s\left(s, \hat{X}(s), e_i\right) + \mathcal{V}_x\left(s, \hat{X}(s), e_i\right) \left(\hat{X}(s)\, r_0(s) + \hat{M}(s)^\top \tilde{B}(s,e_i) - \hat{c}(s)\right) \\ & + \frac{1}{2}\mathcal{V}_{xx}\left(s, \hat{X}(s), e_i\right) \hat{M}(s)^\top \widetilde{\sigma}(s,e_i)\, \widetilde{\sigma}(s,e_i)^\top \hat{M}(s) + \sum_{j=1}^{D} g_{ij}\left[\mathcal{V}\left(s,\hat{X}(s),e_j\right) - \mathcal{V}\left(s,\hat{X}(s),e_i\right)\right] \Bigg\} \Bigg) ds \\ & + \Im(T-t) \sum_{j\neq i} \left\{\mathcal{V}\left(s,\hat{X}(s),e_j\right) - \mathcal{V}\left(s,\hat{X}(s),e_i\right)\right\} dJ^{ij}(s) \\ & + \Im(T-t)\, \mathcal{V}_x\left(s,\hat{X}(s),\alpha(s)\right) \hat{M}(s)^\top \widetilde{\sigma}(s,\alpha(s))\, dW^\star(s). \end{aligned} \tag{4.10}$$

Next, comparing the ds term in (4.10) by the ones in the second equation in (4.6), we deduce that

$$\begin{aligned} & \mathcal{V}_s\left(s,\hat{X}(s),e_i\right) + \mathcal{V}_x\left(s,\hat{X}(s),e_i\right)\left(\hat{X}(s)\, r_0(s) + \hat{M}(s)^\top \tilde{B}(s,e_i) - \hat{c}(s)\right) \\ & + \frac{1}{2}\mathcal{V}_{xx}\left(s,\hat{X}(s),e_i\right) \hat{M}(s)^\top \widetilde{\sigma}(s,e_i)\, \widetilde{\sigma}(s,e_i)^\top \hat{M}(s) \\ & + \sum_{j=1}^{D} g_{ij}\left(\mathcal{V}\left(s,\hat{X}(s),e_j\right) - \mathcal{V}\left(s,\hat{X}(s),e_i\right)\right) = -r_0(s)\, \mathcal{V}\left(s,\hat{X}(s),e_i\right), \end{aligned} \tag{4.11}$$

and by comparing the $dW^\star(s)$ and dJ^{ij} terms, we also obtain

$$\begin{aligned} q(s,t) &= \Im(T-t)\, \mathcal{V}_x\left(s,\hat{X}(s),\alpha(s)\right) \widetilde{\sigma}(s,\alpha(s))^\top \hat{M}(s), \\ l_{ij}(s,t) &= \Im(T-t)\left\{\mathcal{V}\left(s,\hat{X}(s),e_j\right) - \mathcal{V}\left(s,\hat{X}(s),e_i\right)\right\}. \end{aligned} \tag{4.12}$$

We take the above expressions of $p(s;t)$ and $q(s;t)$ at $s=t$ into (4.7) and (4.8), then

$$\Im(T-t)\, \mathcal{V}\left(t,\hat{X}(t),\alpha(t)\right) - \vartheta_x(\hat{c}(t)) = 0, \tag{4.13}$$

and

$$\mathcal{V}_x\left(t,\hat{X}(t),\alpha(t)\right) \widetilde{\sigma}(t,\alpha(t))\, \widetilde{\sigma}(t,\alpha(t))^\top \hat{M}(t) = -\tilde{B}(t,\alpha(t))\, \mathcal{V}\left(t,\hat{X}(t),\alpha(t)\right). \tag{4.14}$$

Consequently, we get that $\hat{c}(\cdot), \hat{a}(\cdot)$ and $\hat{\pi}(\cdot)$ admit the following representation:

$$\hat{c}(t) = \mathcal{I}\left(\Im(T-t)\,\mathcal{V}\left(t, \hat{X}(t), \alpha(t)\right)\right),\ dt - a.e., \tag{4.15}$$

$$\hat{a}(t) = -\frac{\theta(\alpha(t))\, m_Y \mathcal{V}\left(t, \hat{X}(t), \alpha(t)\right)}{\sigma_Y \mathcal{V}_x\left(t, \hat{X}(t), \alpha(t)\right)},\ dt - a.e., \tag{4.16}$$

$$\hat{\pi}(t) = -\frac{r(t, \alpha(t))\,\mathcal{V}\left(t, \hat{X}(t), \alpha(t)\right)}{\sigma(t, \alpha(t))\,\sigma(t, \alpha(t))^{\top}\,\mathcal{V}_x\left(t, \hat{X}(t), \alpha(t)\right)},\ dt - a.e. \tag{4.17}$$

Then by putting expressions $(4.15), (4.16)$ and (4.17) into (4.11), this indicates that $\mathcal{V}(\cdot,\cdot,\cdot)$ coincides with the solution of the PDE (4.1), evaluated along the trajectory $\hat{X}(\cdot)$, solution of the state equation. ■

Remark 11 *Equation* (4.1) *is comparable with the one in Pirvu & Zhang [42], in which the equilibrium is defined within the class of feedback controls.*

5 Special utility functions

In this section, we look at some special cases of Merton's portfolio problem with general discounting in which the function $\text{ß}(\cdot,\cdot,\cdot)$ may be separated into functions of time and state variables. The equilibrium strategies can then be determined completely by solving a system of ODEs.

5.1 Power utility function

To explicitly solve the problems $(2.11) - (2.12)$, we consider the case where $\vartheta(c) = \dfrac{c^{\gamma}}{\gamma}$ and $h(x) = a\dfrac{x^{\gamma}}{\gamma}$, with $a > 0$ and $\gamma \in (0,1)$. The PDE (4.1) is reduced in this case to

$$\begin{cases} -\text{ß}_t(t,x,e_i) = \text{ß}_x(t,x,e_i)\left(r_0(t)\,x - \tilde{B}(t,e_i)^{\top}\Sigma(t,e_i)\,\tilde{B}(t,e_i)\dfrac{\text{ß}(t,x,e_i)}{\text{ß}_x(t,x,e_i)}\right. \\ \qquad\qquad \left. -\left(\Im(T-t)\,\text{ß}(t,x,e_i)\right)^{\frac{1}{\gamma-1}}\right) + \dfrac{1}{2}\text{ß}_{xx}(t,x,e_i)\,\tilde{B}(t,e_i)^{\top}\Sigma(t,e_i)\,\tilde{B}(t,e_i)\left(\dfrac{\text{ß}(t,x,e_i)}{\text{ß}_x(t,x,e_i)}\right)^2 \\ \qquad\qquad + r_0(t)\,\text{ß}(t,x,e_i) + \sum\limits_{j=1}^{D} g_{ij}\left[\text{ß}(t,x,e_j) - \text{ß}(t,x,e_i)\right],\ \ (t,x,e_i) \in [0,T]\times\mathbb{R}\times\mathcal{X}, \\ \text{ß}(T,x,e_i) = a x^{\gamma-1}. \end{cases} \tag{5.1}$$

We consider the following trial solution based on the terminal condition $\text{ß}(s,x,e_i) = a\Pi(s,e_i)\,x^{\gamma-1}$, for some deterministic function $\Pi(\cdot,e_i) \in C^1([0,T],\mathbb{R})$ for each $e_i \in \mathcal{X}$ where the terminal condition $\Pi(T,e_i) = 1$. Then by substituting in (5.1), we get

$$\begin{cases} \Pi_t(t,e_i) + \left(K(t,e_i) + Q(t)\,\Pi(t,e_i)^{\frac{1}{\gamma-1}}\right)\Pi(t,e_i) + \sum\limits_{j\neq i}^{D} g_{ij}\Pi(t,e_j) = 0,\ \text{for } t \in [0,T], \\ \Pi(T,e_i) = 1, \end{cases} \tag{5.2}$$

where

$$K(t,e_i) \equiv \gamma r_0(t) + \frac{1}{2}\frac{\gamma}{(1-\gamma)}\tilde{B}(t,e_i)^{\top}\Sigma(t,e_i)\,\tilde{B}(t,e_i) + g_{ii}, \tag{5.3}$$

and

$$Q(t) \equiv (1-\gamma)\left(a\Im(T-t)\right)^{\frac{1}{\gamma-1}}. \tag{5.4}$$

From Pirvu & Zhang [42], we deduce that the equation (5.1) admit a unique continuously differentiable uniformly bounded solution $\Pi(t,e_i), e_i \in \mathcal{X}$, then in light of Theorem 10, the representation of the Nash equilibrium strategies (4.3) – (4.5) gives

$$\hat{c}(t) = \left(a\Im\left(T-t\right)\Pi\left(t,\alpha\left(t\right)\right)\right)^{\frac{1}{\gamma-1}}\hat{X}\left(t\right), \ dt-a.e., \tag{5.5}$$

$$\hat{a}\left(t\right) = \frac{\theta\left(\alpha\left(t\right)\right)m_Y}{\left(1-\gamma\right)\sigma_Y}\hat{X}\left(t\right), \ dt-a.e., \tag{5.6}$$

$$\hat{\pi}(t) = \frac{r\left(t,\alpha\left(t\right)\right)}{\left(1-\gamma\right)\sigma\left(t,\alpha\left(t\right)\right)\sigma\left(t,\alpha\left(t\right)\right)^{\top}}\hat{X}\left(t\right), \ dt-a.e. \tag{5.7}$$

The wealth process determined by the above trading strategy is given by

$$\begin{aligned} X\left(t\right) = \quad & x_0 + \int_0^t \left\{ r_0\left(s\right) + \frac{1}{\left(1-\gamma\right)}\tilde{B}\left(s,\alpha\left(s\right)\right)^{\top}\Sigma\left(s,\alpha\left(s\right)\right)\tilde{B}\left(s,\alpha\left(s\right)\right) \right. \\ & \left. -\left(a\Im\left(T-s\right)\Pi\left(s,\alpha\left(s\right)\right)\right)^{\frac{1}{\gamma-1}} \right\} \hat{X}\left(s\right)ds \\ & + \int_0^t \frac{\hat{X}\left(s\right)}{\left(1-\gamma\right)}\tilde{B}\left(s,\alpha\left(s\right)\right)^{\top}\Sigma\left(s,\alpha\left(s\right)\right)\tilde{\sigma}\left(s,\alpha\left(s\right)\right)dW^{\star}\left(s\right), \ t\in\left[0,T\right]. \end{aligned}$$

5.2 Logarithmic utility function

Now, let us consider the case where $\vartheta\left(c\right) = \ln\left(c\right)$ and $h\left(x\right) = a\ln\left(x\right)$, with $a>0$. In this case, the PDE (4.1) reduces to

$$\begin{cases} -\text{ß}_t\left(t,x,e_i\right) = \text{ß}_x\left(t,x,e_i\right)\left(r_0\left(t\right)x - \tilde{B}\left(t,e_i\right)^{\top}\Sigma\left(t,e_i\right)\tilde{B}\left(t,e_i\right)\dfrac{\text{ß}\left(t,x,e_i\right)}{\text{ß}_x\left(t,x,e_i\right)}\right. \\ \qquad \left. -\left(\Im\left(T-t\right)\text{ß}\left(t,x,e_i\right)\right)^{-1}\right) + \dfrac{1}{2}\text{ß}_{xx}\left(t,x,e_i\right)\tilde{B}\left(t,e_i\right)^{\top}\Sigma\left(t,e_i\right)\tilde{B}\left(t,e_i\right)\left(\dfrac{\text{ß}\left(t,x,e_i\right)}{\text{ß}_x\left(t,x,e_i\right)}\right)^2 \\ \qquad + r_0\left(t\right)\text{ß}\left(t,x,e_i\right) + \sum\limits_{j=1}^{D} g_{ij}\left[\text{ß}\left(t,x,e_j\right) - \text{ß}\left(t,x,e_i\right)\right], \ \left(t,x,e_i\right)\in\left[0,T\right]\times\mathbb{R}\times\mathcal{X}, \\ \text{ß}\left(T,x,e_i\right) = \dfrac{a}{x}. \end{cases} \tag{5.8}$$

We consider the ansatz

$$\text{ß}\left(t,x,e_i\right) = \Pi\left(t,e_i\right)\frac{a}{x}, \ \text{for } t\in\left[0,T\right], \tag{5.9}$$

where $\Pi\left(\cdot,e_i\right)\in C^1\left(\left[0,T\right],\mathbb{R}\right)$ for each $e_i\in\mathcal{X}$. Substituting in (5.8), we get

$$\begin{cases} \Pi_t\left(t,e_i\right) + g_{ii}\Pi\left(t,e_i\right) + \dfrac{1}{a\Im\left(T-t\right)} + \sum\limits_{j\neq i}^{D} g_{ij}\Pi\left(t,e_j\right) = 0, \ \text{for } t\in\left[0,T\right], \\ \Pi\left(T,e_i\right) = 1, \end{cases} \tag{5.10}$$

which admits the folowing representation

$$\Pi\left(t,e_i\right) = e^{\left(T-t\right)g_{ii}}\left\{1 + \int_t^T e^{\left(\tau-T\right)g_{ii}}\left(\frac{1}{a\Im\left(T-\tau\right)} + \sum_{j\neq i}^{D} g_{ij}\Pi\left(\tau,e_j\right)\right)d\tau\right\}, \ \text{for } t\in\left[0,T\right]. \tag{5.11}$$

Let $\mathbf{\Pi}_t\left(t\right) = \left(\Pi_t\left(t,e_1\right),.....,\Pi_t\left(t,e_N\right)\right)^T$ and $\mathbf{\Pi}\left(t\right) = \left(\Pi\left(t,e_1\right),....,\Pi\left(t,e_N\right)\right)^T$, the system (5.10) can be represented as

$$\begin{cases} \mathbf{\Pi}_t\left(t\right) + \mathcal{G}\mathbf{\Pi}\left(t\right) + \dfrac{1}{a\Im\left(T-t\right)} = 0, \\ \mathbf{\Pi}\left(T\right) = I\in\mathbb{R}^N, \end{cases}$$

it is well-known that

$$\mathbf{\Pi}\left(t\right) = \left(I + \int_t^T e^{-\left(T-s\right)\mathcal{G}}\frac{1}{a\Im\left(T-s\right)}ds\right)e^{\mathcal{G}\left(T-t\right)}.$$

where the matrix exponential $e^{\mathcal{G}t}$ is defined by

$$e^{\mathcal{G}t}=\sum_{k=0}^{\infty}\frac{(\mathcal{G}t)^k}{k!}.$$

In light of Theorem 10, the representation of the Nash equilibrium strategies (4.3) – (4.5) gives

$$\hat{c}(t)=\left(a\Im\left(T-t\right)\Pi\left(t,\alpha\left(t\right)\right)\right)^{-1}\hat{X}\left(t\right),\ dt-a.e., \tag{5.12}$$

$$\hat{a}\left(t\right)=\frac{\theta\left(\alpha\left(t\right)\right)m_Y}{\sigma_Y}\hat{X}\left(t\right),\ dt-a.e., \tag{5.13}$$

$$\hat{\pi}(t)=\frac{r\left(t,\alpha\left(t\right)\right)}{\sigma\left(t,\alpha\left(t\right)\right)\sigma\left(t,\alpha\left(t\right)\right)^{\top}}\hat{X}\left(t\right),\ dt-a.e. \tag{5.14}$$

The wealth process associated to the above trading strategy is given by

$$\begin{aligned} X\left(t\right) &= x_0+\int_0^t\left\{r_0\left(s\right)+\tilde{B}\left(s,\alpha\left(s\right)\right)^{\top}\Sigma\left(s,\alpha\left(s\right)\right)\tilde{B}\left(s,\alpha\left(s\right)\right)-\left(a\Im\left(T-s\right)\Pi\left(s,\alpha\left(s\right)\right)\right)^{-1}\right\}\hat{X}\left(s\right)ds \\ &\quad +\int_0^t\tilde{B}\left(s,\alpha\left(s\right)\right)^{\top}\Sigma\left(s,\alpha\left(s\right)\right)\widetilde{\sigma}\left(s,\alpha\left(s\right)\right)\hat{X}\left(s\right)dW^{\star}\left(s\right), \end{aligned}$$

where $\forall\left(t,e_i\right)\in[0,T]\times\mathcal{X}$, $\Pi\left(t,e_i\right)$ is given by (5.11).

6 Special discount function

As mentioned in [35], an agent who makes a decision at time t is known as the t-agent and he has the ability to act in two ways: naive and sophisticated. Naive agents take decisions without considering that their preferences will change in the near future, then any t-agent solves the problem on the grounds that it is a standard optimal control problem with initial condition $X(t)=x_t$ such that his decision will be time-inconsistent. The t-agent should then be sophisticated in order to get a time consistent strategy, that is, taking into consideration the preferences of all s-agents, for $s\in[t,T]$. As a result, one way to deal with time-inconsistency in dynamic decision-making problems is to consider them as non-cooperative games with a continuous number of players in which decisions are selected at every instant of time. The solution to the problem of the agent with non-constant discounting must be created by searching for the sub-game perfect equilibria of the related game with an infinite number of t-agents. In [35] the authors looked for a solution of sophisticated agent to the modified HJB, then they must consider Markov equilibrium strategies. Unlike [35], we use open-loop equilibrium strategies in our work. This is a significant difference that leads to a significant shift in the results.

6.1 Exponential discounting with constant discount rate (classical model)

We start this subsection with the case where the discount function is of standard exponential form

$$\Im\left(t\right)=e^{-\delta_0 t},\quad t\in[0,T], \tag{6.1}$$

where $\delta_0>0$ is a constant denotes the discount rate. In this case, our equilibrium solutions for the two cases become

1) **Logarithmic utility**

$$\hat{c}(t)=\left(ae^{-(T-t)\delta_0}\Pi\left(t,\alpha\left(t\right)\right)\right)^{-1}\hat{X}\left(t\right),dt-a.e., \tag{6.2}$$

$$\hat{a}\left(t\right)=\frac{\theta\left(\alpha\left(t\right)\right)m_Y}{\sigma_Y}\hat{X}\left(t\right),\ dt-a.e., \tag{6.3}$$

$$\hat{\pi}(t)=\frac{r\left(t,\alpha\left(t\right)\right)}{\sigma\left(t,\alpha\left(t\right)\right)\sigma\left(t,\alpha\left(t\right)\right)^{\top}}\hat{X}\left(t\right),\ dt-a.e. \tag{6.4}$$

where $\forall\left(t,e_i\right)\in[0,T]\times\mathcal{X}$

$$\begin{cases}\Pi_t\left(t,e_i\right)+g_{ii}\Pi\left(t,e_i\right)+\dfrac{1}{ae^{-(T-t)\delta_0}}+\sum\limits_{j\neq i}^{D}g_{ij}\Pi\left(t,e_j\right)=0,\ \text{for } t\in[0,T],\\ \Pi\left(T,e_i\right)=1,\end{cases}\tag{6.5}$$

Thus we have the following solution

$$\begin{aligned}\mathbf{\Pi}(t) &= \left(\Pi\left(t,e_1\right),....,\Pi\left(t,e_N\right)\right),\\ &= \left(I+\int_t^T e^{-(T-s)\mathcal{G}}\frac{1}{ae^{-(T-s)\delta_0}}ds\right)e^{(T-t)\mathcal{G}}.\end{aligned}$$

2) **Power utility**

$$\hat{c}(t)=\left(ae^{-(T-t)\delta_0}\Pi\left(t,\alpha(t)\right)\right)^{\frac{1}{\gamma-1}}\hat{X}(t),\ dt-a.e.,\tag{6.6}$$

$$\hat{a}(t)=\frac{\theta\left(\alpha(t)\right)m_Y}{(1-\gamma)\sigma_Y}\hat{X}(t),\ dt-a.e.,\tag{6.7}$$

$$\hat{\pi}(t)=\frac{r\left(t,\alpha(t)\right)}{(1-\gamma)\sigma\left(t,\alpha(t)\right)\sigma\left(t,\alpha(t)\right)^{\top}}\hat{X}(t),\ dt-a.e,\tag{6.8}$$

where $\forall\left(t,e_i\right)\in[0,T]\times\mathcal{X}$, $\Pi\left(t,e_i\right)$ is the solution to equation (5.2), with $Q(t)=(1-\gamma)\left(a\exp\left(-\delta_0(T-t)\right)\right)^{\frac{1}{\gamma-1}}$.

6.2 Exponential discounting with non constant discount rate (Karp's model)

Let us now suppose that the instantaneous discount rate is non-constant, which is a continuous and positive function of time $\delta(l)$, for $l\in[0,T]$, as proposed by Karp [28]. A non-increasing discount rate $\delta(\cdot)$ will characterize impatient agents. To evaluate a payoff at time $\tau\geq 0$, the given discount factor is given by $\Im(\tau)=e^{-\int_0^{\tau}\delta(l)dl}$. In this case, our (open-loop) equilibrium solutions for the two cases become

1) **Logarithmic utility**

$$\hat{c}(t)=\left(ae^{-\int_0^{T-t}\delta(l)dl}\Pi\left(t,\alpha(t)\right)\right)^{-1}\hat{X}(t),dt-a.e.,\tag{6.10}$$

$$\hat{a}(t)=\frac{\theta\left(\alpha(t)\right)m_Y}{\sigma_Y}\hat{X}(t),\ dt-a.e.,\tag{6.11}$$

$$\hat{\pi}(t)=\frac{r\left(t,\alpha(t)\right)}{\sigma\left(t,\alpha(t)\right)\sigma\left(t,\alpha(t)\right)^{\top}}\hat{X}(t),\ dt-a.e.,\tag{6.12}$$

where $\forall\left(t,e_i\right)\in[0,T]\times\mathcal{X}$

$$\begin{cases}\Pi_t\left(t,e_i\right)+g_{ii}\Pi\left(t,e_i\right)+\dfrac{1}{ae^{-\int_0^{T-t}\delta(l)dl}}+\sum\limits_{j\neq i}^{D}g_{ij}\Pi\left(t,e_j\right)=0,\ \text{for } t\in[0,T],\\ \Pi\left(T,e_i\right)=1,\end{cases}\tag{6.13}$$

Thus we have the following solution

$$\begin{aligned}\mathbf{\Pi}(t) &= \left(\Pi\left(t,e_1\right),....,\Pi\left(t,e_N\right)\right),\\ &= \left(I+\int_t^T e^{-(T-s)\mathcal{G}}\frac{1}{ae^{-\int_0^{T-s}\delta(l)dl}}ds\right)e^{(T-t)\mathcal{G}}.\end{aligned}$$

2) **Power utility**

$$\hat{c}(t)=\left(ae^{-\int_0^{T-t}\delta(l)dl}\Pi\left(t,\alpha(t)\right)\right)^{\frac{1}{\gamma-1}}\hat{X}(t),\ dt-a.e.,\tag{6.14}$$

$$\hat{a}(t)=\frac{\theta\left(\alpha(t)\right)m_Y}{(1-\gamma)\sigma_Y}\hat{X}(t),\ dt-a.e.,\tag{6.15}$$

$$\hat{\pi}(t)=\frac{r\left(t,\alpha(t)\right)}{(1-\gamma)\sigma\left(t,\alpha(t)\right)\sigma\left(t,\alpha(t)\right)^{\top}}\hat{X}(t),\ dt-a.e,\tag{6.16}$$

where $\forall\,(t,e_i)\in[0,T]\times\mathcal{X}, \Pi\,(t,e_i)$ is the solution to equation (5.2) with $Q\,(t)=(1-\gamma)\left(a\exp\left(-\int_0^{T-t}\delta_0\,(l)\,dl\right)\right)^{\frac{1}{\gamma-1}}$.

6.3 Hyperbolic discounting

We conclude this subsection with the case where the discount function is of hyperbolic form, which induces dynamically inconsistent preferences, implying a motive for consumers to constrain their own future choices. Hyperbolic discounting is mathematically described as

$$\Im\,(\tau)=\frac{1}{1+\delta\tau} \tag{6.17}$$

where $\Im\,(\cdot)$ is the discount factor and $\delta>0$ is a constant representing the discount rate. In this case, our equilibrium solutions for the two cases become

1) **Logarithmic utility**

$$\hat{c}(t)=\left(\frac{a}{1+(T-t)\,\delta}\Pi\,(t,\alpha\,(t))\right)^{-1}\hat{X}\,(t)\,,dt-a.e., \tag{6.18}$$

$$\hat{a}\,(t)=\frac{\theta\,(\alpha\,(t))\,m_Y}{\sigma_Y}\hat{X}\,(t)\,,\ dt-a.e., \tag{6.19}$$

$$\hat{\pi}(t)=\frac{r\,(t,\alpha\,(t))}{\sigma\,(t,\alpha\,(t))\,\sigma\,(t,\alpha\,(t))^{\top}}\hat{X}\,(t)\,,\ dt-a.e., \tag{6.20}$$

where $\forall\,(t,e_i)\in[0,T]\times\mathcal{X}$

$$\begin{cases}\Pi_t\,(t,e_i)+g_{ii}\Pi\,(t,e_i)+\dfrac{1+(T-t)\,\delta}{a}+\sum\limits_{j\neq i}^{D}g_{ij}\Pi\,(t,e_j)=0,\ \text{for } t\in[0,T]\,,\\ \Pi\,(T,e_i)=1,\end{cases} \tag{6.21}$$

Thus we have the following solution

$$\begin{aligned}\mathbf{\Pi}\,(t)&=(\Pi\,(t,e_1)\,,....,\Pi\,(t,e_N))\,,\\ &=\left(I+\int_t^T\frac{1+(T-s)\,\delta}{a}e^{-(T-s)\mathcal{G}}ds\right)e^{(T-t)\mathcal{G}}.\end{aligned}$$

2) **Power utility**

$$\hat{c}(t)=\left(\frac{a}{1+(T-t)\,\delta}\Pi\,(t,\alpha\,(t))\right)^{\frac{1}{\gamma-1}}\hat{X}\,(t)\,,\ dt-a.e., \tag{6.22}$$

$$\hat{a}\,(t)=\frac{\theta\,(\alpha\,(t))\,m_Y}{(1-\gamma)\,\sigma_Y}\hat{X}\,(t)\,,\ dt-a.e., \tag{6.23}$$

$$\hat{\pi}(t)=\frac{r\,(t,\alpha\,(t))}{(1-\gamma)\,\sigma\,(t,\alpha\,(t))\,\sigma\,(t,\alpha\,(t))^{\top}}\hat{X}\,(t)\,,\ dt-a.e. \tag{6.24}$$

where $\forall\,(t,e_i)\in[0,T]\times\mathcal{X}$, $\Pi\,(t,e_i)$ is the solution to equation (5.2) with $Q\,(t)=(1-\gamma)\left(\dfrac{a}{1+\delta\,(T-t)}\right)^{\frac{1}{\gamma-1}}$.

7 Numerical analysis

In this section, we present some numerical results to illustrate the effects of model parameters on the results derived in the previous section. Throughout the numerical analyses for convenience but without loss of generality we consider the logarithmic utility case where the discount function is of hyperbolic form. Other cases can be treated in a similar manner, we suppose that all the parameters of the financial market are constants and the Markov chain takes two possible values 1 and 2, i.e., $\mathcal{X}=\{1,2\}$ with the rate matrix of the Markov chain being

$$\mathcal{G}=\begin{pmatrix}-2 & 2\\ 5 & -5\end{pmatrix}.$$

the basic parameters are given below:

	a	$\theta(\alpha(t))$	δ_0	λ_L	m_Y	σ_Y	δ	r_0	$\mu(\alpha(t))$	$\sigma(\alpha(t))$
$\alpha(t)=1$	0.5	0.8	1.5	0.5	0.3	0.4	0.6	0.35	0.7	0.3
$\alpha(t)=2$	0.5	0.6	1.5	0.5	0.3	0.4	0.6	0.35	0.6	0.4

For $(t,i)\in[0,T]\times\{1,2\}$, let us denote by $\mathbf{A}(t,i)$, $\mathbf{C}(t,i)$ and $\boldsymbol{\pi}(t,i)$ the propensities to equilibrium reinsurance, consumption and investment strategies, respectively, i.e. $\mathbf{A}(t,\alpha(t))=\frac{\hat{a}(t)}{\hat{X}(t)}$, $\mathbf{C}(t,\alpha(t))=\frac{\hat{c}(t)}{\hat{X}(t)}$ and $\boldsymbol{\pi}(t,\alpha(t))=\frac{\hat{\pi}(t)}{\hat{X}(t)}$.

According to (6.19), we can obtain that $\dfrac{\partial\mathbf{A}}{\partial\theta(\cdot)}(t,\alpha(t))>0$, $\dfrac{\partial\mathbf{A}}{\partial m_Y}(t,\alpha(t))>0$ and $\dfrac{\partial\mathbf{A}}{\partial\sigma_Y}(t,\alpha(t))<0$ which indicate that as the safety loading $\theta(\cdot)$ or the expectation of the size of each claim m_Y increases, the reinsurance becomes more expensive, thus the insurer will undertake more risk through purchasing less reinsurance or acquiring more new business. When the claims' second-order moment σ_Y is higher, i.e. the surplus becomes more volatile, the insurer will purchase more reinsurance or acquire less new business.

Figure 1 presents the curves of the different state trajectories of the propensitie to equilibrium reinsurance $\mathbf{A}(t,i)$, in the three mods, $i=1$, $i=2$ and $i=\alpha(t)$. By using a two-state Markov chain Matlab code for $\alpha(\cdot)$, we can achieve the trajectories of $\mathbf{A}(t,1)$, $\mathbf{A}(t,2)$ and $\mathbf{A}(t,\alpha(t))$ and their graphs, the blue line is the graph of $\mathbf{A}(t,1)$, the continuous red line is the graph of $\mathbf{A}(t,2)$, and the solid black line is the graph of $\mathbf{A}(t,\alpha(t))$, whose values are switched between the blue line and the red line.

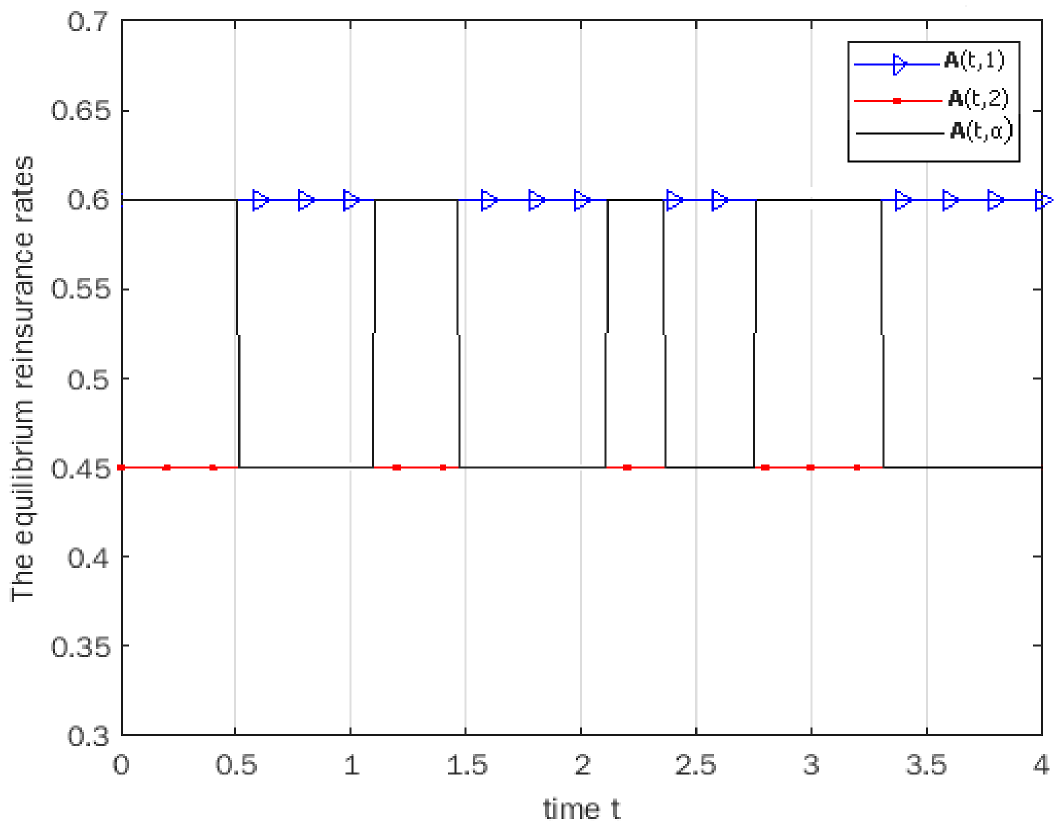


Figure 1. The propensitie to equilibrium reinsurance in the three modes for i=1,2 and α.

From the expression (6.20), we know that it is only dependent with the parameters of risk-free asset and risky asset. Differentiating $\boldsymbol{\pi}(t,\alpha(t))$ with respect to r_0, we have $\dfrac{\partial\boldsymbol{\pi}}{\partial r_0}(t,\alpha(t))<0$, which implies that the cost of borrowing and lending will be higher as the interest rate increases. Thus, the insurer should invest less money in the risky asset. We obtain that $\dfrac{\partial\boldsymbol{\pi}}{\partial\mu}(t,\alpha(t))>0$, which shows that the equilibrium investment strategy increases with μ. In other words, the insurer will invest more money in the risky asset with the increase of μ,

$\dfrac{\partial \boldsymbol{\pi}}{\partial \sigma}(t, \alpha(t)) < 0$, which shows $\boldsymbol{\pi}(t, \alpha(t))$ decrease with the volatility of the risky asset's price. The insurer should reduce investment in the risky asset when σ becomes larger to hedge the risk.

Figure 2 depicts the graph of the propensitie to equilibrium investment with respect to the state change of the Markov chain $\alpha(\cdot)$ between 0 and 4 unit of time, where the initial state is assume to be $\alpha(0) = 1$.

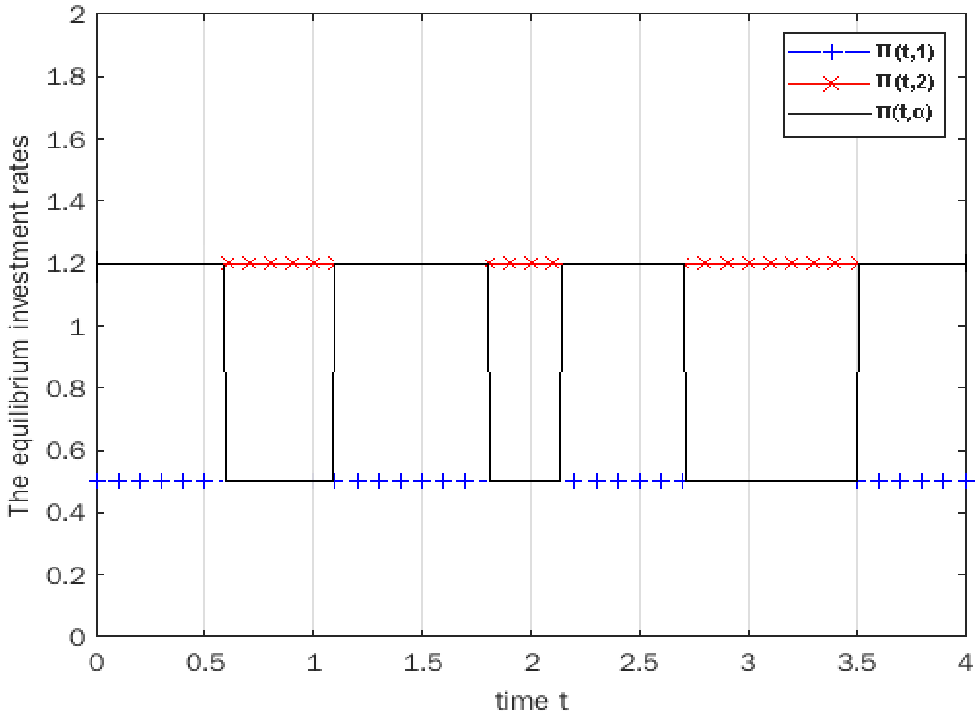
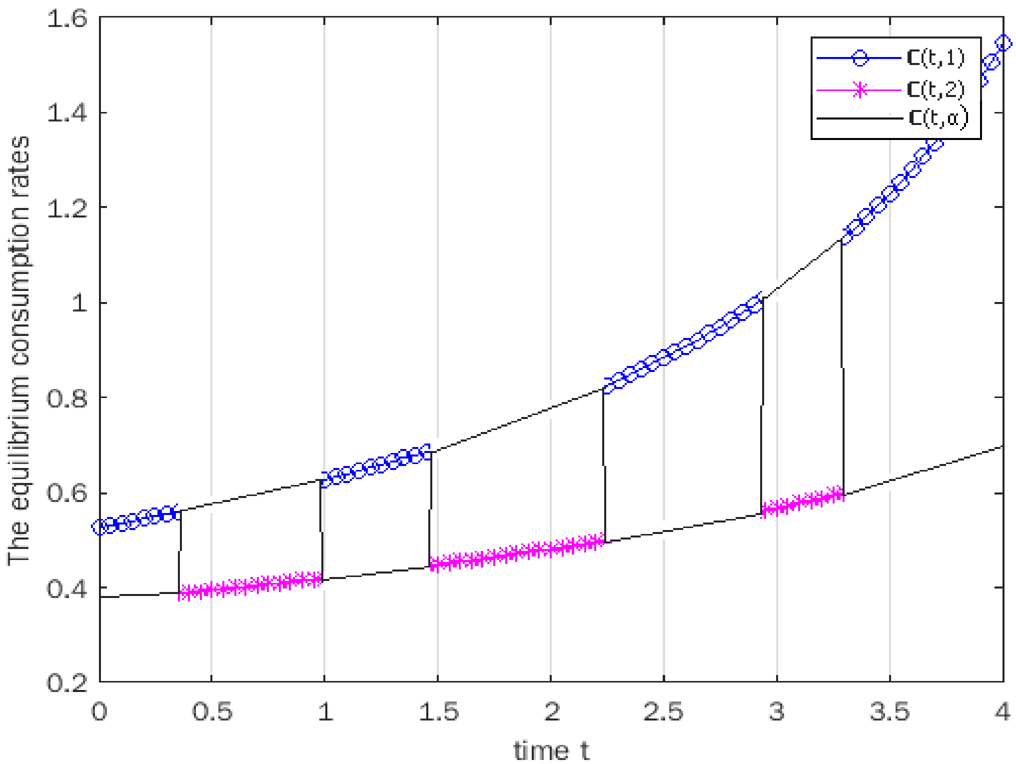


Figure 2. The propensitie to equilibrium investment in the three modes for i=1,2 and α.

By using Matlab's advanced ODE solvers (particularly the function ode45) and Markov chain $\alpha(\cdot)$ we can achieve trajectories of $C(t,1)$, $C(t,2)$ and $C(t,\alpha(t))$. Figure 3 shows the state of the equilibrium consumption propensitie. In fact, when $\alpha(0) = 1$, is the initial state trajectory. Then the values are also switched between two paths which are the trajectories of the equilibrium consumption propensitie correspond to the different states of the Markov chain $\alpha(t) = 1$ and $\alpha(t) = 2$. As a result, we can clearly see how the Markovian switching influences the overall behaviour of the trajectories of the equilibrium consumption propensitie.



The propensitie to equilibrium consumption in the three modes for i=1,2 and α.

8 Conclusion

In this paper, we have considered a class of dynamic decision models of Merton's consumption-investment and reinsurance problem, under the effect of Markovian regime-switching. We have employed a game theoretic advance to handle the time-inconsistency. Through this study, open-loop Nash equilibrium strategies are established as an alternative of optimal strategies. This was achieved using a stochastic system that includes a flow of forward-backward stochastic differential equations under equilibrium conditions. Concrete instances of discounted utilities are presented to confirm the validity of our proposed study. The work may be developed in different ways. The methodology, for example, may be expanded to a non-Markovian framework meaning that, the case where the coefficients of the controlled SDE as well as the coefficients of the objective functional are random, via closed-loop equilibrium strategies.

Another problem is to consider the model with some constraints such as negativity on the wealth process (see e.g. [40]) and to extend our objective criterion with deterministic discount function to the one with stochastic discount process. It is also worth to discuss some other types of reinsurance such as excess-of-loss reinsurance or combined reinsurance in our risk model.

Similar to the finite-state models, the Markov chain can be assumed to take values infinite state space $\mathbb{N}$, and the rate matrix being $\mathcal{G} = [g_{ij}]_{\mathbb{N}\times\mathbb{N}}$. A direct consequence is that the matrix $\mathcal{G}$ has an infinite dimension, where $i, j \in \mathbb{N}$ and $g_{ij} > 0$. From the definition, each row of $\mathcal{G}$ must sum up to 0, $\sum_{j=1}^{\infty} g_{ij} = 0$. If we impose the non-explosivity condition which is

$$\sup_{i\in\mathbb{N}} -g_{ii} < \infty,$$

then the usual Martingale problem mentioned in first display of Page 3 has a unique solution for the continuous time Markov chain and everything will work in similar fashion like in finite state space. The reason we went ahead with finite state space Markov chain assumption is that the major focus of this work is on applied directions, and finite state space makes all the calculations tractable as opposed to countably infinite state space.

But infinite state space may give some interesting challenges from other aspects when we study the long run behavior of open loop Nash equilibrium strategies in specific examples. Recall that in case of finite state Markov chain if e_i is a positive recurrent state, then its recurrence time T_{e_i} has finite expected value. It turns out that the difficulty is the fact that while every state e_i communicates with every other state e_j, it is possible that the chain starting from e_i wanders off to "infinity" for every without ever returning to e_i (for transient case). Furthermore, it is possible that even if the chain returns to e_i infinitely often with probability 1, the expected return time from e_i to e_i can be infinite (for null recurrent case). Few works done on regime switching process with countably infinite state space are [33] and [46]. Other general references include [48] and [10].

Then how to study the methods where the regime-switching process is allowed to take infinite many states is a very interesting and challenging research problems.

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9 Appendix

Proof of Proposition 6 is derived using some limiting procedures and duality analysis. Furthermore, because our objective function is not in the quadratic form, we have to adapt some of the results obtained in [25] and [26] to our control problem which is about maximizing a general and non necessary quadratic utility. The proof of Lemma 7 follows a similar argument to [23]

Proof of Proposition 6. The estimates (3.15) – (3.17) follow from Theorem 4.4 in [53]. Moreover the following representation regarding the objective function

$$\begin{aligned} & J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon}(\cdot)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right) \\ & =\mathbb{E}^{t}\left[\int_{t}^{T}\Im(s-t)\left(\vartheta\left(\pounds^{\top}u^{\varepsilon}(s)\right)-\vartheta\left(\pounds^{\top}\hat{u}(s)\right)\right)ds+\Im(T-t)\left(h\left(X^{\varepsilon}(T)\right)-h\left(\hat{X}(T)\right)\right)\right]. \end{aligned} \tag{A.1.1}$$

Now, from (3.12) and by applying the second order Taylor-Young expansion, we find that

$$\begin{aligned} h\left(\hat{X}^{\varepsilon}(T)\right)-h\left(\hat{X}(T)\right) &= h_x\left(\hat{X}(T)\right)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)+\frac{1}{2}h_{xx}\left(\hat{X}(T)\right)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^2 \\ &\quad + o\left(\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^2\right). \end{aligned}$$

By applying the second order Taylor-Lagrange expansion we get

$$\vartheta\left(\pounds^{\top}u^{\varepsilon}(s)\right)-\vartheta\left(\pounds^{\top}\hat{u}(s)\right)=\left\langle\vartheta_x\left(\pounds^{\top}\hat{u}(s)\right)\pounds,v\right\rangle+\frac{1}{2}\left\langle\vartheta_{xx}\left(\pounds^{\top}\hat{u}(s)+\lambda v 1_{[t,t+\varepsilon)}\right)\pounds\pounds^{\top}v,v\right\rangle.$$

From (3.17) it holds that

$$\begin{aligned} & J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon}(\cdot)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right) \\ & =\mathbb{E}^{t}\left[\int_{t}^{T}\Im(s-t)\left\{\left\langle\vartheta_x\left(\pounds^{\top}\hat{u}(s)\right)\pounds,v\right\rangle+\frac{1}{2}\left\langle\vartheta_{xx}\left(\pounds^{\top}\hat{u}(s)+\lambda v 1_{[t,t+\varepsilon)}\right)\pounds\pounds^{\top}v,v\right\rangle\right\}1_{[t,t+\varepsilon)}ds\right. \\ & \quad\left.+\Im(T-t)\left(h_x\left(\hat{X}(T)\right)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)+\frac{1}{2}h_{xx}\left(\hat{X}(T)\right)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)^2\right)\right]+o(\varepsilon). \end{aligned} \tag{A.1.2}$$

Notice that

$$\begin{aligned} & \Im(T-t)\left(h_x\left(\hat{X}(T)\right)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)+\frac{1}{2}h_{xx}\left(\hat{X}(T)\right)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)^2\right) \\ & =p(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)+\frac{1}{2}P(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)^2. \end{aligned}$$

Now, by applying Itô's formula to $s \mapsto p(s;t)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)$ on $[t,T]$, we get

$$\mathbb{E}^t\left[p(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)\right]=\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{v^\top B(s,\alpha(s))\,p(s;t)+v^\top D(s,\alpha(s))\,\widetilde{q}(s;t)\right\}ds\right]. \tag{A.1.3}$$

Again, by applying Itô's formula to $s \mapsto P(s;t)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^2$ on $[t,T]$, we get

$$\begin{aligned}&\mathbb{E}^t\left[P(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right)^2\right]\\&=\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{2v^\top\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)\left(B(s,\alpha(s))\,P(s,t)+D(s,\alpha(s))\,\widetilde{Q}(s,t)\right)\right.\right.\\&\qquad\left.\left.+v^\top\left(D(s,\alpha(s))\,D(s,\alpha(s))^\top\right)vP(s,t)\right\}ds\right],\end{aligned} \tag{A.1.4}$$

where $\widetilde{Q}(s;t)=\left(0,Q(s;t)^\top\right)^\top$. On the other hand, we conclude from **(H1)** together with (3.17) that

$$\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)\left(B(s,\alpha(s))\,P(s,t)+D(s,\alpha(s))\,\widetilde{Q}(s,t)\right)ds\right]=o(\varepsilon). \tag{A.1.5}$$

By taking $(A.1.3)$, $(A.1.4)$ and $(A.1.5)$ in $(A.1.2)$, it follows that

$$\begin{aligned}&J\left(t,\hat{X}(t),\alpha(t);u^\varepsilon(\cdot)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right)\\&=\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{\left\langle B(s,\alpha(s))\,p(s;t)+D(s,\alpha(s))\,\widetilde{q}(s;t)+\Im(s-t)\,\vartheta_x\left(\pounds^\top\hat{u}(s)\right)\pounds,v\right\rangle\right.\right.\\&\quad\left.\left.+\frac{1}{2}\left\langle\left(\Im(s-t)\,\vartheta_{xx}\left(\left\langle\pounds,\hat{u}(s)+\lambda v1_{[t,t+\varepsilon)}\right\rangle\right)\pounds\pounds^\top+P(s,t)\,D(s,\alpha(s))\,D(s,\alpha(s))^\top\right)v,v\right\rangle\right\}ds\right]+o(\varepsilon),\end{aligned}$$

which is equivalent to (3.18).

Now, we derive the proof of Lemma 7 by using some limiting procedures. First, let us recall the following lemma which was proved by Wang in [49], Lemma 3.3.

Lemma 12 *If $\phi(\cdot)=(\phi_1(\cdot),...,\phi_m(\cdot))\in\mathcal{M}^p_{\mathcal{F}}(0,T;\mathbb{R}^m)$ with $m\in\mathbb{N}$ and $p>1$, then for a.e. $t\in[0,T)$, there exists a sequence $\{\varepsilon_n^t\}_{n\in\mathbb{N}}\subset(0,T-t)$ depending on t such that $\lim_{n\to\infty}\varepsilon_n^t=0$ and*

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\left|\phi_i(s)-\phi_i(t)\right|^p ds\right]=0,\ \textit{for } i=1,...,m,\ d\mathbb{P}-a.s.$$

Proof of Lemma 7. We define, for $t\in[0,T]$ and $s\in[t,T]$,

$$\left(\bar{p}(s;t),\bar{q}(s;t),\bar{l}(s;t)\right):=\frac{e^{-\int_s^T r_0(\tau)d\tau}}{\Im(T-t)}\left(p(s;t),q(s;t),l(s;t)\right).$$

Then, for any $t\in[0,T]$, in the interval $[t,T]$, the process $\left(\bar{p}(\cdot;t),\bar{q}(\cdot;t),\bar{l}(\cdot;t)\right)$ satisfies

$$\begin{cases}d\bar{p}(s;t)=&\bar{q}(s;t)^\top dW^\star(s)+\sum_{j\neq i}\bar{l}_{ij}(s;t)\,d\Phi^{ij}(s),\ \ s\in[t,T],\\ \bar{p}(T;t)=&h_x\left(\hat{X}(T)\right),\end{cases} \tag{A.2.1}$$

Moreover, it is clear that from the uniqueness of solutions to $(A.2.1)$, we have the equality $\left(\bar{p}(s;t_1),\bar{q}(s;t_1),\bar{l}(s;t_1)\right)=\left(\bar{p}(s;t_2),\bar{q}(s;t_2),\bar{l}(s;t_2)\right)$, for any $t_1,t_2,s\in[0,T]$ such that $0<t_1<t_2<s<T$. Hence, the solution $\left(\bar{p}(\cdot;t),\bar{q}(\cdot;t),\bar{l}(\cdot;t)\right)$ does not depend on the variable t and this allows us to denote the solution of $(A.2.1)$ by $\left(\bar{p}(\cdot),\bar{q}(\cdot),\bar{l}(\cdot)\right)$.

We have then, for any $t\in[0,T]$, and $s\in[t,T]$,

$$\left(p(s;t),q(s;t),l(s;t)\right)=\Im(T-t)\,e^{\int_s^T r_0(\tau)d\tau}\left(\bar{p}(s),\bar{q}(s),\bar{l}(s)\right). \tag{A.2.2}$$

Now using $(A.2.2)$ we have, under **(H2)**, for any $t\in[0,T]$ and $s\in[t,T]$,

$$|p(s;t)-p(s;s)|\leq\sup_{t\leq s\leq t+\varepsilon}|\Im(T-t)-\Im(T-s)|\,e^{-\int_s^T r_0(\tau)d\tau}\,|\bar{p}(s)|, \tag{A.2.3}$$

and

$$|q(s;t)-q(s;s)|\leq\sup_{t\leq s\leq t+\varepsilon}|\Im(T-t)-\Im(T-s)|\,e^{-\int_s^T r_0(\tau)d\tau}\,|\bar{q}(s)|, \tag{A.2.4}$$

From which, we have for any $a>0$, $t\in[0,T]$, and $\varepsilon\in(0,T-t)$,

$$\begin{aligned}&\mathbb{P}\left(\left|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathcal{H}(s;t)\,ds\right]-\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathcal{H}(s;s)\,ds\right]\right|\geq a\right),\\
&\leq\frac{1}{a}\mathbb{E}\left|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathcal{H}(s;t)\,ds\right]-\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathcal{H}(s;s)\,ds\right]\right|,\\
&\leq C\sup_{t\leq s\leq t+\varepsilon}|\Im(T-t)-\Im(T-s)|\,\frac{1}{\varepsilon}\mathbb{E}\int_t^{t+\varepsilon}(|\bar{p}(s)|+|\bar{q}(s)|)\,ds\\
&\quad+\sup_{t\leq s\leq t+\varepsilon}|\Im(s-t)-1|\,\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[\vartheta_x\left(\pounds^\top\hat{u}(s)\right)\right]ds.\end{aligned}$$

Noting that since $\Im(\cdot)$ is continuous we get $\lim\limits_{\varepsilon\downarrow 0}\sup\limits_{t\leq s\leq t+\varepsilon}|\Im(T-t)-\Im(T-s)|=0$ for $t\in[0,T]$. Moreover, since $(\bar{p}(\cdot),\bar{q}(\cdot))\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})\times\mathcal{M}^2_{\mathcal{F}}\left(0,T;\mathbb{R}^{N+1}\right)$ we get

$$\lim_{\varepsilon\downarrow 0}\sup_{t\leq s\leq t+\varepsilon}|\Im(T-t)-\Im(T-s)|\,\frac{1}{\varepsilon}\mathbb{E}\int_t^{t+\varepsilon}(|\bar{p}(s)|+|\bar{q}(s)|)\,ds=0.$$

Noting that $\Im(0)=1$ then $\lim\limits_{\varepsilon\downarrow 0}\sup\limits_{t\leq s\leq t+\varepsilon}|\Im(s-t)-1|=0$. According to **(H3)**, by using the dominated convergence theorem

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[\vartheta_x\left(\pounds^\top\hat{u}(s)\right)\right]ds=\mathbb{E}\left[\vartheta_x\left(\pounds^\top\hat{u}(t)\right)\right]<\infty,\ dt-a.e.$$

Therefore

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathcal{H}(s;t)\,ds\right]-\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathcal{H}(s;s)\,ds\right]\right|=0.$$

Hence, for each t there exists a sequence $(\varepsilon_n^t)_{n\geq 0}\subset(0,T-t)$ such that $\lim\limits_{n\to\infty}\varepsilon_n^t=0$ and

$$\lim_{n\to\infty}\left|\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\mathcal{H}(s;t)\,ds\right]-\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\mathcal{H}(s;s)\,ds\right]\right|=0,\ d\mathbb{P}-a.s.$$

Moreover, since $\vartheta_x\left(\pounds^\top\hat{u}(\cdot)\right)\in\mathcal{M}^p_{\mathcal{F}}(0,T;\mathbb{R})$ and

$$(\bar{p}(\cdot),\bar{q}(\cdot))\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})\times\mathcal{M}^2_{\mathcal{F}}\left(0,T;\mathbb{R}^{N+1}\right)$$

we get from Lemma 12 that, there exists a subsequence of $(\varepsilon_n^t)_{n\geq 0}$ which also denote by $(\varepsilon_n^t)_{n\geq 0}$ such that

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\mathcal{H}(s;s)\,ds\right]=\mathcal{H}(t;t),\ dt-a.e,\ d\mathbb{P}-a.s.$$

To derive the statement 2) in the Lemma 7, it is sufficient to prove the following, for each t there exists a sequence $(\varepsilon_n^t)_{n\geq 0}\subset(0,T-t)$ such that $\lim\limits_{n\to\infty}\varepsilon_n^t=0$ and

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\Im(s-t)\,\vartheta_{xx}\left(\pounds^\top\left(\hat{u}(s)+\lambda v1_{[t,t+\varepsilon)}\right)\right)ds\right]=\vartheta_{xx}\left(\pounds^\top(\hat{u}(t))\right),\\
&\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\widetilde{\sigma}(s,\alpha(s))\,\widetilde{\sigma}(s,\alpha(s))^\top P(s;t)\,ds\right]=\widetilde{\sigma}(t,\alpha(t))\,\widetilde{\sigma}(t,\alpha(t))^\top P(t;t).\end{aligned}$$

Let us prove the first limit. We have

$$\left|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\Im\left(s-t\right)\vartheta_{xx}\left(\pounds^\top\left(\hat{u}\left(s\right)+\lambda v1_{[t,t+\varepsilon)}\right)\right)ds\right]\right.$$
$$\left.-\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\vartheta_{xx}\left(\pounds^\top\left(\hat{u}\left(s\right)\right)\right)ds\right]\right|$$
$$\leq \sup_{t\leq s\leq t+\varepsilon}\left|\Im\left(s-t\right)-1\right|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\sup_{\eta\leq M}\left|\vartheta_{xx}\left(\pounds^\top\left(\hat{u}\left(s\right)+\eta\right)\right)\right|ds\right].$$

Applying the same arguments used in the first limit, we obtain according to Lemma 12,

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\vartheta_{xx}\left(\pounds^\top\hat{u}\left(s\right)\right)ds\right]=\vartheta_{xx}\left(\pounds^\top\hat{u}\left(t\right)\right),$$

at least for a subsequence.

Conflicts of Interest.
The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] G. Ainslie (1995): *Special reward: A behavioral theory of impulsiveness and impulse control.* Psychological Bulletin 82: 463-496.

[2] I. Alia, F. Chighoub and A. Sohail (2016): *A characterization of equilibrium strategies in continuous-time mean variance problems for insurers.* Insurance: Mathematics and Economics 68: 212-223.

[3] I. Alia and F. Chighoub (2021): *Continuous-time mean-variance portfolio selection with regime-switching financial market: Time-consistent solution.* Random Operators and Stochastic Equations 29 (1): 11-25

[4] I. Alia, F. Chighoub, N. Khelfallah and J. Vives (2021): *Time-consistent investment and consumption strategies under a general discount function.* J. Risk Financial Manag., 14, 86. https://doi.org/10.3390/jrfm14020086

[5] L. Bai and H. Zhang (2008): *Dynamic mean-variance problem with constrained risk control for the insurers.* Mathematical Methods of Operations Research 68: 181–205.

[6] R. J. Barro (1990): *Ramsey meets Laibson in the neoclassical growth model.* Quarterly Journal of Economics 114: 1125-1152.

[7] S. Basak and G. Chabakauri (2010): *Dynamic mean-variance asset allocation.* Review of Financial Studies 23: 2970-3016.

[8] T. Björk and A. Murgoci (2008): *A general theory of Markovian time-inconsistent stochastic control problems.* SSRN Electronic Journal 1694759.

[9] T. Björk, A. Murgoci and X. Y. Zhou (2014): *Mean-variance portfolio optimization with state-dependent risk aversion.* Mathematical Finance 24 (1): 1-24.

[10] A. Borodin and V. Gorin (2013): *Markov processes of infinitely many nonintersecting random walks,* Probability Theory and Related Fields 155 (3-4): 935–997.

[11] S. Browne (1995): *Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin.* Mathematics of Operations Research 20: 937–957.

[12] P. Chen, H.Yang And G. Yin (2008): *Markowitz's mean-variance asset-liability management with regime switching: A continuous-time model.* Insurance: Mathematics and Economics 43 (3):456-465.

[13] P. Chen And H. Yang (2011): *Markowitz's mean-variance asset-liability management with regime switching: A multi period model*. Applied Mathematical Finance 18 (1): 29-50.

[14] C. Czichowsky (2013): *Time-consistent mean-variance portfolio selection in discrete and continuous time*. Finance and Stochastics 17 (2): 227-271.

[15] C. Donnelly (2011): *Sufficient stochastic maximum principle in a regime-switching diffusion model*. Applied Mathematics And Optimization 64 (2): 155–169.

[16] I. Ekeland and A. Lazrak (2008): *Equilibrium policies when preferences are time-inconsistent*. ArXiv:0808.3790v1.

[17] I. Ekeland and T. A. Pirvu (2008): *Investment and consumption without commitment*. Mathematics and Financial Economics 2: 57-86.

[18] I. Ekeland, O. Mbodji and T. A. Pirvu (2012): *Time-consistent portfolio management*. SIAM Journal of Financial Mathematics 3: 1-32.

[19] R. J. Elliott, L. Aggoun And J.B. Moore (1994): *Hidden Markov Models: Estimation and Control*. Springer, New York.

[20] W. H. Fleming and T. Zariphopoulou (1991): *An optimal investment/consumption model with borrowing constraints*. Mathematics of Operations Research 16: 802-822.

[21] S. M. Goldman (1980): *Consistent plans*. Review of Financial Studies 47: 533-537.

[22] J. Grandell (1991): *Aspects of risk theory*. New York Springer.

[23] Y. Hamaguchi (2021): *Time-inconsistent consumption-investment problems in incomplete markets under general discount functions.* SIAM Journal on Control and Optimization, 59(3): 2121–2146.

[24] Y. Hamaguchi (2021): *Small-time solvability of a flow of forward–backward stochastic differential equations.* Applied Mathematics & Optimization, 84: 567–588.

[25] Y. Hu, H. Jin and X. Y. Zhou (2012): *Time-inconsistent stochastic linear quadratic control*. SIAM Journal on Control and Optimization 50 (3): 1548-1572.

[26] Y. Hu, H. Jin and X. Y. Zhou (2015): *Time-Inconsistent Stochastic Linear-Quadratic Control: Characterization and Uniqueness of Equilibrium*. SIAM Journal on Control and Optimization 55 (2): 1261-1279.

[27] I. Karatzas, J. Lehoczky and S. E. Shreve (1987): *Optimal portfolio and consumption decisions for a "small investor" on a finite horizon*. SIAM Journal on Control and Optimization 25: 1157-1186.

[28] L. Karp (2007): *Non-constant discounting in continuous time*. Journal of Economic Theory 132: 557-568.

[29] P. Krusell and A. Smith (2003): *Consumption and savings decisions with quasi-geometric discounting*. Econometrica 71: 366-375.

[30] F. E. Kydland and E. Prescott (1997): *Rules rather than discretion: The inconsistency of optimal plans*. Journal of Political Economy 85: 473-492.

[31] Y. Li and H. Zheng (2015): *Weak Necessary and Sufficient Stochastic Maximum Principle for Markovian Regime-Switching Diffusion Models*. Applied Mathematics And Optimization 71 (1): 39–77.

[32] Z. Liang And M. Song (2015): *Time-consistent reinsurance and investment strategies for mean-variance insurer under partial information*. Insurance: Mathematics and Economics 65: 66–76.

[33] Z. W. Liao and J. Shao (2020): Existence of optimal delay-dependent control for finite-horizon continuous-time Markov decision process. arXiv preprint arXiv:2003.13982.

[34] G. Loewenstein and D. Prelec (1992): *Anomalies in inter-temporal choice: Evidence and an interpretation*. The Quarterly Journal of Economics 107 (2): 573-597.

[35] J. Marín-Solano and J. Navas (2010): *Consumption and portfolio rules for time-inconsistent investors*. European Journal of Operational Research 201: 860-872.

[36] R. C. Merton (1969): *Lifetime portfolio selection under uncertainty: the continuous-time case*. Review of Econometrics Statistics 51: 247-257.

[37] R. C. Merton (1971): *Optimum consumption and portfolio rules in a continuous-time model*. Journal of Economic Theory 3: 373-413.

[38] F. J. Mhlanga, L. C. Galane, N. Mwareya, E. Chikodza and C. Guambe (2022): *Stochastic differential game strategies in the presence of reinsurance and dividend payout*, Journal of Industrial and Management Optimization 19(5): 3589–3609.

[39] K. N. C. Njoku and B. O. Osu (2019): *Effect of Inflation on Stochastic Optimal Investment Strategies for DC Pension under the Affine Interest Rate Model*. Fundamental Journal of Mathematics and Applications 2(1): 91-100.

[40] S. Park and B.G. Jang (2014): *Optimal retirement strategy with a negative wealth constraint.* Operations Research Letters, 42(3): 208–212.

[41] E. S. Phelps and R. A. Pollak (1968): *On second-best national saving and game-equilibrium growth.* The Review of Economic Studies 35: 185-199.

[42] T.A. Pirvu and H. Zhang (2014): *Investment–consumption with regime-switching discount rates*. Mathematical Social Sciences 71: 142–150.

[43] R. Pollak (1968): *Consistent planning*. The Review of Financial Studies 35: 185-199.

[44] F. P. Ramsey (1928): *A Mathematical Theory of Saving*. Economic Journal 38: 543-559.

[45] L. C. G. Rogers and D. Williams (2000): *Diffusions, Markov Processes, and Martingales*. Cambridge Math. Libr., Cambridge University, Cambridge (2).

[46] J. Shao (2019): The existence of optimal feedback controls for stochastic dynamical systems with regime-switching. arXiv preprint arXiv:1906.08425.

[47] R. Strotz (1955): *Myopia and inconsistency in dynamic utility maximization*. The Review of Economic Studies 23: 165-180.

[48] H. C. Tijms, M. C.T. Van De Coeverring (1991): *A simple numerical approach for infinite-state Markov chains*, Prob. Engineer. Inform. Sci., 5: 285-295.

[49] T. Wang (2020): *Uniqueness of equilibrium strategies in dynamic mean-variance problems with random coefficients.* Journal of mathematical analysis and applications 490 (1): Article 124199.

[50] J. Wei, K. C. Wong, S. C. P. Yam And S. P. Yung (2013): *Markowitz's mean–variance asset-liability management with regime switching: a time-consistent approach*. Insurance: Mathematics and Economics 53 (1): 281–291.

[51] J. Yong (2011): *A deterministic linear quadratic time-inconsistent optimal control problem*. Mathematical Control and Related Fields 1: 83-118.

[52] J. Yong (2012): *Time-inconsistent optimal control problems and the equilibrium HJB equation*. Mathematical Control and Related Fields 2 (3): 271-329.

[53] J. Yong and X. Y. Zhou (1999): *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer.

[54] Q. Zhao, Y. Shen and J. Wei (2014): *Consumption-investment strategies with non-exponential discounting and logarithmic utility*. European Journal of Operational Research 283 (3): 824-835.

[55] X.Y. Zhou And G. Yin (2003): *Markowitzs mean–variance portfolio selection with regime switching: a continuous-time model*. SIAM Journal on Control and Optimization 42:1466–1482.

[56] Z. Zou, S. Chen and L. Wedge (2014): *Finite horizon consumption and portfolio decisions with stochastic hyperbolic discounting*. Journal of Mathematical Economics 52: 70-80.