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Highlighting some of the issues with multicurrency numéraires

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ABSTRACT

Multicurrency numéraires are weighted baskets of currencies, in which the weights are positive and sum to one. Some popular multicurrency numéraires are: trade-weighted, the IMF's Special Drawing Right (SDR), equally-weighted, risk-weighted, and GDP-weighted. But which multicurrency numéraire should be used to model systems of currencies? This paper highlights some of the issues that arise when using multicurrency numéraires by utilising two conditions, namely, a consistency condition and a no-arbitrage condition. It is shown that both the consistency condition and the no-arbitrage condition are satisfied by common equally-weighted multicurrency numéraires. As a consequence, it is recommended that common equally-weighted multicurrency numéraires be used to model systems of currencies.

1. Introduction

A bilateral exchange rate is the price of a currency in terms of another currency: a single-currency numéraire. In contrast, a multilateral exchange rate is the price of currency in terms of a multicurrency numéraire, which is weighted basket of currencies. Bilateral exchange rates are the standard prices used to model systems of currencies. However, it has been shown that using bilateral exchange rates in the Frankel-Wei regression framework[1] results in a biased estimator of the regression coefficient, which is caused by a common single-currency numéraire appearing in both bilateral exchange rates (Kunkler, 2021). In addition, it is recommended that multilateral exchange rates be used in the Frankel-Wei regression framework (Kunkler, 2021). In this context, which multicurrency numéraire should be used to price multilateral exchange rates? The motivation of this paper is to highlight some of the issues that arise when using different multicurrency numéraires to provide guidance on which multicurrency numéraire to use when modelling currencies.

Multicurrency numéraires are weighted baskets of currencies, where the weights are positive and sum to one. Different types of multicurrency numéraires include: trade-weighted baskets, equally-weighted baskets, risk-weighted baskets, GDP-weighted baskets, Special Drawing Right (SDR) basket, and other weighted baskets. The SDR of the International Monetary Fund is a reserve asset that is used to supplement the reserves of central banks (IMF, 2024). Trade-weighted baskets are used by central banks to measure their local currencies relative to major trading countries (Loretan, 2005). Common equally-weighted baskets that consists of all currencies of interest are used to provide numéraire invariant currency indexes (Hovanov et al., 2004, 2007). In contrast, idiosyncratic equally-weighted baskets have been used for individual currency risk factors, such as the US dollar risk factor (Lustig et al., 2011; Verdelhan, 2018), and in the currency-basket approach (Aloosh and Bekaert, 2022).

This paper contributes to the literature by utilising two conditions to highlight issues that arise when using multicurrency

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[1] See Frankel & Wei (1994).

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numéraires. The two conditions include: a consistency condition and a no-arbitrage condition. It is shown that both the consistency condition and the no-arbitrage condition are satisfied for a common equally-weighted multicurrency numéraire that includes all currencies of interest.

2. Methodology and results

2.1. Multicurrency-numéraire quadrants

Let there be a system of N currencies. Multicurrency numéraires are weighted baskets of currencies, where the weights are positive and sum to one:

$$\sum_{k=1}^{N} w_k = 1 \tag{1}$$

where w_k is the weight of the kth currency, and $w_k \geq 0$. It is assumed that the currencies in a multicurrency numéraire are a subset of the system of N currencies, so some of the weights in Eq. (1) can be zero.

Multicurrency numéraires can be grouped by using two *dichotomous* variables, namely, currency selection and currency weights. The currency selection variable is concerned with the currencies included in a multicurrency numéraire and consists of two values: idiosyncratic and common. The currency weights variable is concerned with the distribution of the weights in a multicurrency numéraire and consists of two values: unequal and equal. Fig. 1 displays the two values of the currency selection variable on the vertical axis and the two values of the currency weights variable on the horizontal axis.[2] As a result, multicurrency numéraires can be classified into four groups or quadrants: an idiosyncratic-unequal quadrant; an idiosyncratic-equal quadrant; a common-unequal quadrant; and a common-equal quadrant.

The *idiosyncratic-unequal quadrant* is the most general, where each currency can be priced in terms of an idiosyncratic (different) unequally-weighted multicurrency numéraire. Examples include trade-weighted multicurrency numéraires, where each currency is priced in terms of its own trade-weighted currency basket.

The *idiosyncratic-equal quadrant* is less general, where each currency can be priced in terms of an idiosyncratic equally-weighted multicurrency numéraire. An example is the individual currency risk factors, such as the US dollar risk factor and the Eurozone euro risk factor (Verdelhan, 2018). Another example is the currency-basket approach (Aloosh and Bekaert, 2022). For individual currency risk factors and the currency-basket approach, each currency is priced in terms of an equally-weighted basket of *other* currencies. In fact, individual currency risk factors and the currency-basket approach are equivalent. For clarity, individual currency risk factors will be used throughout the rest of this paper to represent both.

The *common-unequal quadrant* is more restricted, where each currency is priced in terms of a common multicurrency numéraire with unequal weights. Some examples include: the Special Drawing Right (SDR), GDP-weighted baskets, and risk-weighted baskets.

Finally, the *common-equal quadrant* is the most restricted, where each currency is priced in terms of a common multicurrency numéraire with equal weights. Some examples include numéraire invariant currency indexes (Hovanov, et. al, 2004, 2007) and relative currency rates (Kunkler and MacDonald, 2015).

2.2. Multicurrency numéraire examples

The data sample consists a system of eight ($N = 8$) currencies, including: the US dollar (USD), the Eurozone euro (EUR), the Japanese yen (JPY), the British pound (GBP), the Canadian dollar (CAD), the Swedish krona (SEK), the Swiss franc (CHF), and the Chinese renminbi (CNY). Six multicurrency numéraires are used to represent the four multicurrency-numéraire quadrants.

The *idiosyncratic-unequal quadrant* is represented by two trade-weighted baskets, namely, the ICE US Dollar Index multicurrency numéraire (M_{USD_ICE}) and the ICE Euro Index multicurrency numéraire (M_{EUR_ICE}) from Intercontinental Exchange (ICE) (see ICE, 2021). The ICE US Dollar Index multicurrency numéraire consists of six currencies of the top trading partners of the United States, namely, the Eurozone euro at 57.60 %, the Japanese yen at 13.60 %, the British pound at 11.90 %, the Canadian dollar at 9.10 %, the Swedish krona at 4.20 %, and the Swiss franc at 3.60 %. The weights of the ICE US Dollar Index multicurrency numéraire have been constant since 1973 (ICE, 2021). The ICE Euro Index multicurrency numéraire consists of five currencies of the top trading partners of the Eurozone, namely, the US dollar at 33.70 %, the Japanese yen at 14.30 %, the British pound at 27.60 %, the Swedish krona at 9.70 %, and the Swiss franc at 14.70 %.

The *idiosyncratic-equal quadrant* is represented by two individual currency risk factors, namely, a US dollar risk factor multicurrency numéraire (M_{USD_CRF}), and a Eurozone euro risk factor multicurrency numéraire (M_{EUR_CRF}). Both multicurrency numéraires consist of all eight currencies except the associated principal currency, which creates seven currencies with equal weights of 14.29% $= 1/7$.

The *common-unequal quadrant* is represented by the Special Drawing Right multicurrency numéraire (M_{COM_SDR}), which consists of five currencies, namely, the US dollar at 43.38 %, the Eurozone euro at 29.31 %, the Japanese yen at 7.59 %, the British pound at 7.44 %, and the Chinese renminbi at 12.28 %. The Special Drawing Right (SDR) multicurrency numéraire changes every five years, and the weights above have been effective since the 1st of August 2022 (IMF, 2024).

[2] The style of Fig. 1 was inspired by Figure 4.1 in Beck (1999).

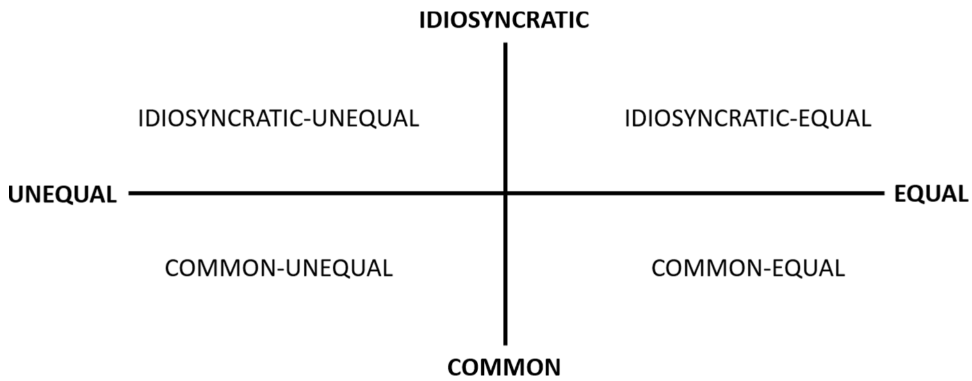

Fig. 1. Multicurrency-numéraire quadrants.

Finally, the *common-equal quadrant* is represented by a common equally-weighted multicurrency numéraire that includes all currencies (M_{COM_EQL}), which consists of all eight currencies with equal weights of 12.50% = 1/8.

Table 1 reports the weights of the different multilateral numéraires. The three non-equally-weighted baskets contain large weights in single currencies. The ICE US Dollar Index multilateral numéraire (M_{USD_ICE}) contains a large weight of 57.60 % in the Eurozone euro (*EUR*), the ICE Euro Index multilateral numéraire (M_{EUR_ICE}) contains a large weight of 33.37 % in the US dollar (*USD*), and the Special Drawing Right (*SDR*) contains a large weight of 41.68 % in the US dollar (*USD*).

In contrast, the weights for the three equally-weighted multicurrency numéraires are evenly distributed across the currencies in the associated multicurrency numéraires. The common equally-weighted multicurrency numéraire (M_{COM_EQL}) has a common weight of 12.50% = 1/8 for all eight currencies. Similarly, the weights for both the US dollar risk factor multicurrency numéraire (M_{USD_CRF}) and the Eurozone euro risk factor multicurrency numéraire (M_{EUR_CRF}) have a common weight of 14.29% = 1/7 for all seven *other* currencies. Note that the principal currency of each currency risk factor is not included in the associated multicurrency numéraire. The US dollar (principal currency) is not included in the US dollar risk factor multicurrency numéraire (M_{USD_CRF}) and the Eurozone euro (principal currency) is not included in the Eurozone euro risk factor multicurrency numéraire (M_{EUR_CRF}).

2.3. Bilateral exchange rates

A bilateral exchange rate is the price of a currency in terms of a single-currency numéraire. In log terms, each bilateral exchange rate in a system of *N* currencies can be written as:

$$p_{i/j} = w_i - w_j = 1_i - 1_j \tag{2}$$

where $i,j = 1,\ldots,N$; $p_{i/j}$ is the *i*th/*j*th bilateral exchange rate; $w_i = 1_i$ is a weight of one worth of the *i*th currency; $w_j = 1_j$ is a weight of one worth of the *j*th (*numéraire*) currency; and $p_{i/i,t} = 0$. For example, the weights associated with buying the Eurozone euro/US dollar (*EUR/USD*) bilateral exchange rate are:

$$p_{EUR/USD} = w_{EUR} - w_{USD} = 1_{EUR} - 1_{USD} \tag{3}$$

where $p_{EUR/USD}$ is the *EUR/USD* bilateral exchange rate; $w_{EUR} = 1_{EUR}$ is a weight of one worth of the Eurozone euro; and $w_{USD} = 1_{USD}$ is a weight of one worth of the US dollar. To buy the *EUR/USD* bilateral exchange rate, an investor buys a weight of one worth of the Eurozone euro (1_{EUR}) and sells a weight of one worth of the US dollar (1_{USD}).

Table 1
Multicurrency numéraires.

Index	Currency Selection	Currency Weights	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
M_{USD_ICE}	Idiosyncratic	Unequal		0.5760	0.1360	0.1190	0.0910	0.0420	0.0360	
M_{EUR_ICE}	Idiosyncratic	Unequal	0.3370		0.1430	0.2760		0.0970	0.1470	
M_{COM_SDR}	Common	Unequal	0.4338	0.2931	0.0759	0.0744				0.1228
M_{USD_CRF}	Idiosyncratic	Equal		0.1429	0.1429	0.1429	0.1429	0.1429	0.1429	0.1429
M_{EUR_CRF}	Idiosyncratic	Equal	0.1429		0.1429	0.1429	0.1429	0.1429	0.1429	0.1429
M_{COM_EQL}	Common	Equal	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250

Notes: Table 1 reports the weights for different multicurrency numéraires, namely, the ICE US Dollar Index (M_{USD_ICE}); the ICE Euro Index (M_{EUR_ICE}); the Special Drawing Right (M_{COM_SDR}); the US dollar risk factor (M_{USD_CRF}); the Eurozone euro risk factor (M_{EUR_CRF}); and the common equally-weighted (M_{COM_EQL}).

2.4. Multilateral exchange rates

A multilateral exchange rate is the price of a currency in terms of a multicurrency numéraire. In general, it is assumed that each currency in a system of N currencies can be priced in terms of an idiosyncratic (different) multicurrency numéraire. For example, the US dollar could be priced in terms of a trade-weighted multicurrency numéraire for the United States and the Eurozone euro could be priced in terms of a trade-weighted multicurrency numéraire for the Eurozone.

In log terms, a multilateral exchange rate can be written as a weighted basket of bilateral exchange rates:

$$p_{i/M_i} = \sum_{k=1}^{N} w_{k,M_i} p_{i/k} \tag{4}$$

where $i = 1,\ldots,N$; p_{i/M_i} is the multilateral exchange rate for the ith currency in terms of the multicurrency numéraire of the ith currency (M_i); w_{k,M_i} is the weight for the kth currency in the multicurrency numéraire of the ith currency (M_i); and $p_{i/k}$ is the ith/kth bilateral exchange rate.

By substituting $p_{i/k} = 1_i - 1_k$ from Eq. (2) into Eq. (4), a multilateral exchange rate can be written as:

$$p_{i/M_i} = \sum_{k=1}^{N} w_{k,M_i}(1_i - 1_k) = 1_i \sum_{k=1}^{N} w_{k,M_i} - \sum_{k=1}^{N} w_{k,M_i} 1_k = 1_i - \overline{w}_{M_i} \tag{5}$$

where $i = 1,\ldots,N$; p_{i/M_i} is the multilateral exchange rate for the ith currency in terms of the multicurrency numéraire for the ith currency (M_i); 1_i is a weight of one worth of the ith currency; $\sum_{k=1}^{N} w_{k,M_i} = 1$ from Eq. (1);

$$\overline{w}_{M_i} = \sum_{k=1}^{N} w_{k,M_i} 1_k \tag{6}$$

is the set of weights of the multicurrency numéraire for the ith currency (M_i); w_{k,M_i} is the weight of the kth currency in the multicurrency numéraire for the ith currency (M_i); and 1_k is a weight of one worth of the kth currency.

2.5. US dollar multilateral exchange rates

Table 2 reports the weights from buying the US dollar multilateral exchange rate in terms of different multicurrency numéraires,

Table 2
US dollar multilateral exchange rates.

	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
Panel A: ICE US Dollar Index								
1_{USD}	1.0000							
$\overline{w}_{M_{USD\_ICE}}$		0.5760	0.1360	0.1190	0.0910	0.0420	0.0360	
$p_{USD/M_{USD\_ICE}}$	1.0000	−0.5760	−0.1360	−0.1190	−0.0910	−0.0420	−0.0360	
Panel B: Special Drawing Right								
1_{USD}	1.0000							
$\overline{w}_{M_{COM\_SDR}}$	0.4338	0.2931	0.0759	0.0744				0.1228
$p_{USD/M_{COM\_SDR}}$	0.5662	−0.2931	−0.0759	−0.0744				−0.1228
Panel C: US dollar risk factor								
1_{USD}	1.0000							
$\overline{w}_{M_{USD\_CRF}}$		0.1429	0.1429	0.1429	0.1429	0.1429	0.1429	0.1429
$p_{USD/M_{USD\_CRF}}$	1.0000	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429
Panel D: Common equally-weighted								
1_{USD}	1.0000							
$\overline{w}_{M_{COM\_EQL}}$	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250
$p_{USD/M_{COM\_EQL}}$	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250

Notes: Table 2 reports the weights from buying the US dollar multilateral exchange rates in terms of four multicurrency numéraires, namely, the ICE US Dollar Index (M_{USD_ICE}) in Panel A, the Special Drawing Right (M_{COM_SDR}) in Panel B, the US dollar risk factor (M_{USD_CRF}) in Panel C, and the common equally-weighted (M_{COM_EQL}) in Panel D.

namely, the ICE US Dollar Index multicurrency numéraire (M_{USD_ICE}), the common SDR multicurrency numéraire (M_{COM_SDR}), the US dollar risk factor multicurrency numéraire (M_{USD_CRF}), and the common equally-weighted multicurrency numéraire (M_{COM_EQL}).

The US dollar multilateral exchange rate in terms of the US dollar risk factor multicurrency numéraire is:

$$\begin{aligned} p_{USD/M_{USD\_CRF}} &= \sum_{k=1,k\neq USD}^{8}\left(w_{k,M_{USD\_CRF}}\times p_{USD/k}\right) \\ &= 1_{USD}-\sum_{k=1,k\neq USD}^{8}\left(\frac{1}{7}\right)_k \\ &= 1_{USD}-\frac{1}{7_{EUR}}-\frac{1}{7_{JPY}}-\frac{1}{7_{GBP}}-\frac{1}{7_{CAD}}-\frac{1}{7_{SEK}}-\frac{1}{7_{CHF}}-\frac{1}{7_{CNY}} \end{aligned} \tag{7}$$

where $p_{USD/M_{USD\_CRF}}$ is the US dollar multilateral exchange rate in terms of the US dollar risk factor multicurrency numéraire (M_{USD_CRF}); 1_{USD} is a weight of one worth of the US dollar; $w_{k,M_{USD\_CRF}}$ is a weight of $1/7$ worth of the kth currency in the US dollar risk factor multicurrency numéraire (M_{USD_CRF}); and $p_{USD/k}$ is the *USD/k*th bilateral exchange rate (Verdelhan, 2018). To buy the US dollar multilateral exchange rate in terms of the US dollar risk factor multicurrency numéraire (M_{USD_CRF}), an investor buys a weight of one worth of the US dollar and sells a weight of $1/7$ worth of each the *other* seven currencies. Note that the US dollar multilateral exchange rate in Eq. (7) does not include the US dollar bilateral exchange rate with itself ($p_{USD/USD}$), so the average is of the *other* seven currencies.

In contrast, the US dollar multilateral exchange rate in terms of the common equally-weighted multicurrency numéraire is:

$$\begin{aligned} p_{USD/M_{COM\_EQL}} &= \sum_{k=1}^{8}\left(w_{k,M_{COM\_EQL}}\times p_{USD/k}\right) \\ &= 1_{USD}-\sum_{k=1}^{N}\left(\frac{1}{8}\right)_k \\ &= \frac{7}{8_{USD}}-\frac{1}{8_{EUR}}-\frac{1}{8_{JPY}}-\frac{1}{8_{GBP}}-\frac{1}{8_{CAD}}-\frac{1}{8_{SEK}}-\frac{1}{8_{CHF}}-\frac{1}{8_{CNY}} \end{aligned} \tag{8}$$

where $p_{USD/M_{COM\_EQL}}$ is the US dollar multilateral exchange rate in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}); 1_{USD} is a weight of one worth of the US dollar; $w_{k,COM\_EQL} = (1/8)_k$ is a weight of $1/8$ worth of the kth currency in the common equally-weighted multicurrency numéraire (M_{COM_EQL}); and $p_{USD/k}$ is the *USD/k*th bilateral exchange rate. To buy the US dollar multilateral exchange rate in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}), an investor buys a weight of $7/8 = (1-1/8)$ worth of the US dollar and sells a weight of $1/8$ worth of each the other seven currencies. Note that the US dollar multilateral exchange rate in Eq. (8) includes the US dollar bilateral exchange rate with itself ($p_{USD/USD}$), so the average is of *all* eight currencies.

2.6. The consistency condition

The consistency condition is satisfied for a system of multilateral exchange rates if the difference between any two multilateral exchange rates is equal to the associated bilateral exchange rate. When the consistency condition is satisfied, the multicurrency numéraires of each multilateral exchange rate cancel with each other.

In log terms, the consistency condition is written as:

$$p_{i/M_i}-p_{j/M_j}=p_{i/j}=1_i-1_j \tag{9}$$

where $i,j = 1,\dots,N$; p_{i/M_i} is the multilateral exchange rate for the ith currency in terms of the ith multicurrency numéraire (M_i); p_{j/M_j} is the multilateral exchange rate for the jth currency in terms of the jth multicurrency numéraire (M_j); and $p_{i/j} = 1_i - 1_j$ is the ith/jth bilateral exchange rate from Eq. (2); 1_i is a weight of one worth of the ith currency; and 1_j is a weight of one worth of the jth currency.

By using Eq. (2), the difference between two multilateral exchange rates in Eq. (9) can be written as:

$$\begin{aligned} p_{i/M_i}-p_{j/M_j} &= \left(1_i-\sum_{k=1}^{N}w_{k,M_i}1_k\right)-\left(1_j-\sum_{k=1}^{N}w_{k,M_j}1_k\right) \\ &= \left(1_i-1_j\right)-\sum_{k=1}^{N}\left(w_{k,M_i}-w_{k,M_j}\right)1_k \\ &= p_{i/j}-\sum_{k=1}^{N}\left(w_{k,M_i}-w_{k,M_j}\right)1_k \end{aligned} \tag{10}$$

where $i,j = 1,\dots,N$; p_{i/M_i} is the multilateral exchange rate for the ith currency in terms of the multicurrency numéraire for the ith currency (M_i); p_{j/M_j} is the multilateral exchange rate for the jth currency in terms of the multicurrency numéraire for the jth currency

(M_j); w_{k,M_i} is the weight of the kth currency in the multicurrency numéraire for the ith currency (M_i); w_{k,M_j} is the weight of the kth currency in the multicurrency numéraire for the jth currency (M_j); $p_{i/j} = 1_i - 1_j$ is the ith/jth bilateral exchange rate from Eq. (2); and 1_i, 1_j, and 1_k are weights of one worth of the ith, jth, and kth currencies, respectively.

The consistency condition in Eq. (9) can be subtracted from Eq. (10) to give:

$$\sum_{k=1}^{N} \left(w_{k,M_i} - w_{k,M_j} \right) 1_k = \sum_{k=1}^{N} 0_k \tag{11}$$

where w_{k,M_i} is the weight of the kth currency in the multicurrency numéraire for the ith currency (M_i); w_{k,M_j} is the weight of the kth currency in the multicurrency numéraire for the jth currency (M_j); 1_k is a weight of one worth of the kth currency; and 0_k is a weight of zero worth of the kth currency. The solution to Eq. (11) is:

$$w_{k,M_i} = w_{k,M_j} \tag{12}$$

for all $i,j = 1, \ldots, N$.

Thus, a system of N multilateral exchange rates satisfy the consistency condition when the weights are equal for each currency in all multicurrency numéraires. This occurs for common multicurrency numéraires in either the common-unequal quadrant or the common-equal quadrant.

Table 3 reports the weights from buying the Eurozone euro multilateral exchange rate and selling the US dollar multilateral exchange rate in terms of different multicurrency numéraires. The consistency condition is satisfied for both common multicurrency numéraires, namely, the common Special Drawing Right multicurrency numéraire (M_{COM_SDR}) and the common equally-weighted multicurrency numéraire (M_{COM_EQL}). For example, the when the common Special Drawing Right multicurrency numéraire (M_{COM_SDR}) is used to price the Eurozone euro and the US dollar, the consistency condition in Eq. (9) can be written as:

$$p_{EUR/M_{EUR\_SDR}} - p_{USD/M_{USD\_SDR}} = 1_{EUR} - 1_{USD} \tag{13}$$

where $p_{EUR/M_{EUR\_SDR}}$ is the Eurozone euro multilateral exchange rate in terms of the common SDR multicurrency numéraire (M_{COM_SDR}); and $p_{USD/M_{USD\_SDR}}$ is the US Dollar multilateral exchange rate in terms of the common SDR multicurrency numéraire (M_{COM_SDR}).

In contrast, the consistency condition is not satisfied for idiosyncratic multicurrency numéraires in either the idiosyncratic-unequal quadrant or the idiosyncratic-equal quadrant. For example, when the ICE trade-weighted multicurrency numéraires are used to price the Eurozone euro and the US dollar, the consistency condition in Eq. (9) can be written as:

Table 3
The consistency condition.

	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
Panel A: Trade-weighted								
$p_{EUR/M_{EUR\_ICE}}$	−0.3370	1.0000	−0.1430	−0.2760	0.0000	−0.0970	−0.1470	
$p_{USD/M_{USD\_ICE}}$	1.0000	−0.5760	−0.1360	−0.1190	−0.0910	−0.0420	−0.0360	
$p_{EUR/USD}$	−1.3370	1.5760	−0.0070	−0.1570	0.0910	−0.0550	−0.1110	
Panel B: Special Drawing Right								
	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
$p_{EUR/M_{COM\_SDR}}$	−0.4338	0.7069	−0.0759	−0.0744				−0.1228
$p_{USD/M_{COM\_SDR}}$	0.5662	−0.2931	−0.0759	−0.0744				−0.1228
$p_{EUR/USD}$	−1.0000	1.0000						
Panel C: Currency risk factors								
	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
$p_{EUR/M_{EUR\_CRF}}$	−0.1429	1.0000	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429
$p_{USD/M_{USD\_CRF}}$	1.0000	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429
$p_{EUR/USD}$	−0.8571	0.8571						
Panel D: Common equally-weighted								
	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
$p_{EUR/M_{COM\_EQL}}$	−0.1250	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250
$p_{USD/M_{COM\_EQL}}$	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250
$p_{EUR/USD}$	−1.0000	1.0000						

Notes: Table 3 reports the weights for buying the Eurozone euro multilateral exchange rate in terms of four multicurrency numéraires and selling the US dollar multilateral exchange rate in terms of the associated four multicurrency numéraires. The multicurrency numéraires are: the ICE Euro Index (M_{EUR_ICE}) and the ICE US Dollar Index (M_{USD_ICE}) in Panel A, the Special Drawing Right (M_{COM_SDR}) in Panel B, the Eurozone euro risk factor (M_{EUR_CRF}) and the US dollar risk factor (M_{USD_CRF}) in Panel C, and the common equally-weighted (M_{COM_EQL}) in Panel D.

$$p_{EUR/M_{EUR\_ICE}} - p_{USD/M_{USD\_ICE}} = -1.337_{USD} + 1.576_{EUR} - 0.007_{JPY} - 0.157_{GBP} + 0.091_{CAD} - 0.055_{SEK} - 0.111_{CHF} \neq 1_{EUR} - 1_{USD} \tag{14}$$

where $p_{EUR/M_{EUR\_SDR}}$ is the Eurozone euro multilateral exchange rate in terms of the ICE Euro Index multicurrency numéraire (M_{EUR_ICE}); and $p_{USD/M_{USD\_SDR}}$ is the US Dollar multilateral exchange rate in terms of the ICE US Dollar Index multicurrency numéraire (M_{USD_ICE}). Not only are the weights for the Eurozone euro (*EUR*) and the US dollar (*USD*) incorrect, but there are significant weights in the other currencies: a negative weight of − 0.157 in the British pound (*GBP*) and a negative weight of − 0.111 in the Swiss franc (*CHF*).

Similarly, the consistency condition is not satisfied for the difference between the Eurozone euro multilateral exchange rate and US multilateral exchange rate when priced in terms of their respective individual currency risk factor multicurrency numéraires, where the consistency condition in Eq. (9) can be written as:

$$p_{EUR/M_{EUR\_CRF}} - p_{USD/M_{USD\_CRF}} = 0.857_{EUR} - 0.857_{USD} \neq 1_{EUR} - 1_{USD} \tag{15}$$

where $p_{EUR/M_{EUR\_CRF}}$ is the Eurozone euro multilateral exchange rate in terms of the Eurozone euro risk factor multicurrency numéraire (M_{EUR_CRF}); and $p_{USD/M_{USD\_CRF}}$ is the US dollar multilateral exchange rate in terms of the US dollar risk factor multicurrency numéraire (M_{USD_CRF}). The individual currency risk factor multicurrency numéraires do not include the principal currency. For instance, the US dollar risk factor multicurrency numéraire (M_{USD_CRF}) does not include the US dollar (principal currency). As a consequence, the multilateral exchanges priced in terms of individual currency risk factor multicurrency numéraires do not satisfy the consistency condition.

2.7. The no-arbitrage condition

The no-arbitrage condition is satisfied for a system of multilateral exchange rates when the sum of all multilateral exchange rates has zero weights in all currencies. In log terms, the no-arbitrage condition is written as:

$$\sum_{i=1}^{N} p_{i/M_i} = \sum_{i=1}^{N} 0_i \tag{16}$$

where p_{i/M_i} is the multilateral exchange rate for the *i*th currency in terms of the multicurrency numéraire of the *i*th currency (M_i); and 0_i is a weight of zero worth of the *i*th currency (see Kunkler and MacDonald, 2015).

The system of *N* multilateral exchange rates in Eq. (5) can be substituted into the summation on the left-hand side of Eq. (16) to give:

$$\sum_{i=1}^{N} p_{i/M_i} = \sum_{i=1}^{N} \left(1_i - \sum_{k=1}^{N} w_{k,M_i} 1_k\right) = \sum_{i=1}^{N} 1_i - \sum_{k=1}^{N} 1_k \sum_{i=1}^{N} w_{k,M_i} \tag{17}$$

where p_{i/M_i} is the multilateral exchange rate for the *i*th currency in terms of the multicurrency numéraire of the *i*th currency (M_i); 1_i is a weight of one worth of the *i*th currency; w_{k,M_i} is the weight of the *k*th currency in the multicurrency numéraire of the *i*th currency (M_i); and 1_k is a weight of one worth of the *k*th currency.

The no-arbitrage condition in Eq. (16) can be subtracted from Eq. (17) to give:

$$\sum_{k=1}^{N} 1_k \sum_{i=1}^{N} w_{k,M_i} = \sum_{k=1}^{N} 1_k \tag{18}$$

where 1_k is a weight of one worth of the *k*th currency; and w_{k,M_i} is the weight of the *k*th currency in the multicurrency numéraire of the *i*th currency (M_i). The solution to Eq. (18) is:

$$\sum_{i=1}^{N} w_{k,M_i} = 1 \tag{19}$$

where $k = 1,\ldots,N$; and w_{k,M_i} is the weight of the *k*th currency in the multicurrency numéraire of the *i*th currency (M_i). Note that the sum in Eq. (19) is across the *N* multicurrency numéraires.

Table 4 reports the weights from buying the system of multilateral exchange rates associated with three multicurrency numéraires, namely, the common SDR multicurrency numéraire (M_{COM_SDR}), the individual currency risk factor multicurrency numéraires (M_{CUR_CRF}), and the common equally-weighted multicurrency numéraire (M_{COM_EQL}).

The no-arbitrage condition is not satisfied for the common SDR multicurrency numéraire (M_{COM_SDR}), where the sum of the multilateral exchange rates has non-zero weights for all numéraire currencies. In contrast, the no-arbitrage condition is satisfied for the equally-weighted multicurrency numéraires. For example, the sum of the multilateral exchange rates in terms of the individual currency risk factor multicurrency numéraires has zero weights for all numéraire currencies. Similarly, the sum of the multilateral exchange rates in terms of the common equally-weighted multicurrency numéraire has zero weights for all numéraire currencies.

2.8. Common equally-weighted multicurrency numéraire

In general, a multilateral exchange rate in terms of any multicurrency numéraire can be written in terms of the system of N multilateral exchange rates in terms of the common equally-weighted multicurrency numéraire by:

$$p_{i/M_i} = p_{i/M_{COM\_EQL}} - \sum_{k=1}^{N} \left(w_{k,M_i} \times p_{k/M_{COM\_EQL}} \right) \tag{20}$$

where $i = 1,\ldots,N$; p_{i/M_i} is the multilateral exchange rate for the ith currency in terms of the ith multicurrency numéraire (M_i); $p_{i/M_{COM\_EQL}}$ is the multilateral exchange rate for the ith currency in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}); w_{k,M_i} is the weight of the kth currency in the multicurrency numéraire of the ith currency (M_i); and $p_{k/M_{COM\_EQL}}$ is the multilateral exchange rate for the kth currency in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}).

Table 5 reports the US dollar multilateral exchange rates from Table 2 written in terms of the system of eight ($N = 8$) multilateral exchange rates in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}). For example, the US dollar multilateral exchange rate in terms of the common SDR multicurrency numéraire can be written as:

$$\begin{aligned} p_{USD/M_{COM\_SDR}} &= p_{USD/M_{COM\_EQL}} - \sum_{k=1}^{N} \left(w_{k,M_{COM\_SDR}} \times p_{k/M_{COM\_EQL}} \right) \\ &= 0.5662_{USD} - 0.2931_{EUR} - 0.0759_{JPY} - 0.0744_{GBP} - 0.1228_{CNY} \end{aligned} \tag{21}$$

where $p_{USD/M_{SDR}}$ is the US dollar multilateral exchange rate in terms of the common *SDR* multicurrency numéraire (M_{COM_SDR}); $p_{USD/M_{COM\_EQL}}$ is the US dollar multilateral exchange rate in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}); $w_{k,M_{COM\_SDR}}$ is the weight of the kth currency in the common *SDR* multicurrency numéraire (M_{COM_SDR}); and $p_{k/M_{COM\_EQL}}$ is the

Table 4
The no-arbitrage condition.

Panel A: Special Drawing Right								
	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
$p_{USD/M_{COM\_SDR}}$	0.5662	−0.2931	−0.0759	−0.0744				−0.1228
$p_{EUR/M_{COM\_SDR}}$	−0.4338	0.7069	−0.0759	−0.0744				−0.1228
$p_{JPY/M_{COM\_SDR}}$	−0.4338	−0.2931	0.9241	−0.0744				−0.1228
$p_{GBP/M_{COM\_SDR}}$	−0.4338	−0.2931	−0.0759	0.9256				−0.1228
$p_{CNY/M_{COM\_SDR}}$	−0.4338	−0.2931	−0.0759	−0.0744				0.8772
Total	−1.1690	−0.4655	0.6205	0.6280				0.3860
Panel B: Currency risk factors								
	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
$p_{USD/M_{USD\_CRF}}$	1.0000	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429
$p_{EUR/M_{EUR\_CRF}}$	−0.1429	1.0000	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429
$p_{JPY/M_{JPY\_CRF}}$	−0.1429	−0.1429	1.0000	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429
$p_{GBP/M_{GBP\_CRF}}$	−0.1429	−0.1429	−0.1429	1.0000	−0.1429	−0.1429	−0.1429	−0.1429
$p_{CAD/M_{CAD\_CRF}}$	−0.1429	−0.1429	−0.1429	−0.1429	1.0000	−0.1429	−0.1429	−0.1429
$p_{SEK/M_{SEK\_CRF}}$	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	1.0000	−0.1429	−0.1429
$p_{CHF/M_{CHF\_CRF}}$	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	1.0000	−0.1429
$p_{CNY/M_{CNY\_CRF}}$	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	1.0000
Total	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Panel C: Common equally-weighted								
	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
$p_{USD/M_{COM\_EQL}}$	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250
$p_{EUR/M_{COM\_EQL}}$	−0.1250	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250
$p_{JPY/M_{COM\_EQL}}$	−0.1250	−0.1250	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250
$p_{GBP/M_{COM\_EQL}}$	−0.1250	−0.1250	−0.1250	0.8750	−0.1250	−0.1250	−0.1250	−0.1250
$p_{CAD/M_{COM\_EQL}}$	−0.1250	−0.1250	−0.1250	−0.1250	0.8750	−0.1250	−0.1250	−0.1250
$p_{SEK/M_{COM\_EQL}}$	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	0.8750	−0.1250	−0.1250
$p_{CHF/M_{COM\_EQL}}$	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	0.8750	−0.1250
$p_{CNY/M_{COM\_EQL}}$	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	0.8750
Total	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Notes: Table 4 reports the weights from buying all multilateral exchange rates priced in terms of three multicurrency numéraires, namely, the common Special Drawing Right (M_{COM_SDR}) in Panel A, the idiosyncratic currency risk factors (M_{CUR_CRF}) in Panel B, and the common equally-weighted (M_{COM_EQL}) in Panel C.

multilateral exchange rate for the kth currency in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}).

In summary, a multilateral exchange rate in terms of any multicurrency numéraire can be written in terms of the system of N multilateral exchange rates in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}). This provides transparency on the underlying positions and risk of any multilateral exchange rate.

3. Limitations

A limitation of this paper is that it does not account for transaction costs in the no-arbitrage condition. For example, the calculation of the daily valuations of the Special Drawing Right (SDR) use mid-market rates, which are the average of bid and offer bilateral exchange rates (IMF, 2024). Another limitation of this paper is that it does not account for time variation in multicurrency numéraires, such as changes that occur in both currency selection and currency weights. For example, the Special Drawing Right (SDR) multicurrency numéraire changes every five years (IMF, 2024). In addition, trade-weighted multicurrency numéraires would also be expected to change as trading patterns change (Loretan, 2005). Both limitations are beyond the scope of this paper and are left for future research.

4. Conclusion

Multicurrency numéraires are weighted basket of currencies. By using two *dichotomous* variables of *currency selection* and *currency weights*, multicurrency numéraires can be classified into four quadrants, namely, idiosyncratic-unequal; idiosyncratic-equal; common-unequal; and common-equal. The currencies are typically weighted, such as trade-weighted, equally-weighted, risk-weighted, GDP-weighted, and others.

This paper highlighted some of the issues that arise when using multicurrency numéraires. A consistency condition and a no-

Table 5
US dollar multilateral exchange rates.

	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
Panel A: ICE US Dollar Index								
$p_{USD/M_{COM\_EQL}}$	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250
$-\ 0.5760\ \times p_{EUR/M_{COM\_EQL}}$	0.0720	−0.5040	0.0720	0.0720	0.0720	0.0720	0.0720	0.0720
$-\ 0.1360\ \times p_{JPY/M_{COM\_EQL}}$	0.0170	0.0170	−0.1190	0.0170	0.0170	0.0170	0.0170	0.0170
$-\ 0.1190\ \times p_{GBP/M_{COM\_EQL}}$	0.0149	0.0149	0.0149	−0.1041	0.0149	0.0149	0.0149	0.0149
$-\ 0.0910\ \times p_{CAD/M_{COM\_EQL}}$	0.0114	0.0114	0.0114	0.0114	−0.0796	0.0114	0.0114	0.0114
$-\ 0.0420\ \times p_{SEK/M_{COM\_EQL}}$	0.0053	0.0053	0.0053	0.0053	0.0053	−0.0368	0.0053	0.0053
$-\ 0.0360\ \times p_{CHF/M_{COM\_EQL}}$	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045	−0.0315	0.0045
$p_{USD/M_{USD\_ICE}}$	1.0000	−0.5760	−0.1360	−0.1190	−0.0910	−0.0420	−0.0360	0.0000
Panel B: Special Drawing Right								
	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
$p_{USD/M_{COM\_EQL}}$	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250
$-\ 0.4338\ \times p_{USD/M_{COM\_EQL}}$	−0.3796	0.0542	0.0542	0.0542	0.0542	0.0542	0.0542	0.0542
$-\ 0.2931\ \times p_{EUR/M_{COM\_EQL}}$	0.0366	−0.2565	0.0366	0.0366	0.0366	0.0366	0.0366	0.0366
$-\ 0.0759\ \times p_{JPY/M_{COM\_EQL}}$	0.0095	0.0095	−0.0664	0.0095	0.0095	0.0095	0.0095	0.0095
$-\ 0.0744\ \times p_{GBP/M_{COM\_EQL}}$	0.0093	0.0093	0.0093	−0.0651	0.0093	0.0093	0.0093	0.0093
$-\ 0.1228\ \times p_{CNY/M_{COM\_EQL}}$	0.0154	0.0154	0.0154	0.0154	0.0154	0.0154	0.0154	−0.1075
$p_{USD/M_{COM\_SDR}}$	0.5662	−0.2931	−0.0759	−0.0744	0.0000	0.0000	0.0000	−0.1228
Panel C: US dollar risk factor								
	USD	EUR	JPY	GBP	CAD	SEK	CHF	CNY
$p_{USD/M_{COM\_EQL}}$	0.8750	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250	−0.1250
$-\ 0.1429\ \times p_{EUR/M_{COM\_EQL}}$	0.0179	−0.1250	0.0179	0.0179	0.0179	0.0179	0.0179	0.0179
$-\ 0.1429\ \times p_{JPY/M_{COM\_EQL}}$	0.0179	0.0179	−0.1250	0.0179	0.0179	0.0179	0.0179	0.0179
$-\ 0.1429\ \times p_{GBP/M_{COM\_EQL}}$	0.0179	0.0179	0.0179	−0.1250	0.0179	0.0179	0.0179	0.0179
$-\ 0.1429\ \times p_{CAD/M_{COM\_EQL}}$	0.0179	0.0179	0.0179	0.0179	−0.1250	0.0179	0.0179	0.0179
$-\ 0.1429\ \times p_{SEK/M_{COM\_EQL}}$	0.0179	0.0179	0.0179	0.0179	0.0179	−0.1250	0.0179	0.0179
$-\ 0.1429\ \times p_{CHF/M_{COM\_EQL}}$	0.0179	0.0179	0.0179	0.0179	0.0179	0.0179	−0.1250	0.0179
$-\ 0.1429\ \times p_{CNY/M_{COM\_EQL}}$	0.0179	0.0179	0.0179	0.0179	0.0179	0.0179	0.0179	−0.1250
$p_{USD/M_{USD\_CRF}}$	1.0000	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429	−0.1429

Notes: Table 5 reports the weights from buying the US dollar multilateral exchange rates in terms of the other multicurrency numéraires written in terms of the system of eight multilateral exchange rates in terms of the common equally-weighted multicurrency numéraire (M_{COM_EQL}). The three multicurrency numéraires are: the ICE US Dollar Index multicurrency numéraire (M_{USD_ICE}) in Panel A, the Special Drawing Right multicurrency numéraire (M_{COM_SDR}) in Panel B, and the US dollar risk factor multicurrency numéraire (M_{USD_CRF}) in Panel C.

arbitrage condition were utilised to provide transparency on the resulting system of multilateral exchange rates. It was shown that the consistency condition and the no-arbitrage condition were both satisfied by a system of multilateral exchange rates in terms of a common equally-weighted multicurrency numéraire. Thus, common equally-weighted multicurrency numéraires are recommended as the multicurrency numéraire of choice to model systems of currencies. Future empirical research is required to measure the significance of using multicurrency numéraires that satisfy the conditions compared to multicurrency numéraires that don't satisfy the conditions.

Disclosure statement

The author reports that there are no competing interests to declare.

Author declaration

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We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

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CRediT authorship contribution statement

Michael Kunkler: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Data availability

Data will be made available on request.

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