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Schatten class Hankel operators on doubling Fock spaces and the Berger-Coburn phenomenon

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ABSTRACT

Using the notion of integral distance to analytic functions, we give a characterization of Schatten class Hankel operators acting on doubling Fock spaces on the complex plane and use it to show that for $f \in L^\infty$, if H_f is Hilbert-Schmidt, then so is $H_{\bar{f}}$. This property is known as the Berger-Coburn phenomenon. When $0 < p \le 1$, we show that the Berger-Coburn phenomenon fails for a large class of doubling Fock spaces. Along the way, we illustrate our results for the canonical weights $|z|^m$ when $m > 0$.

1. Introduction and main results

Let $dA = \frac{1}{2i} dz \wedge d\bar{z}$ be the Lebesgue measure on $\mathbb{C}$, and ϕ be a subharmonic function. For $0 < p < \infty$, $L^p_\phi = L^p(\mathbb{C}, e^{-p\phi} dA)$ is the space of all measurable functions on $\mathbb{C}$ such that

$$\|f\|^p_{p,\phi} = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} dA(z) < \infty, \tag{1.1}$$

and L^∞_ϕ is the space of measurable functions f such that

$$\|f\|_{\infty,\phi} = \operatorname*{ess\,sup}_{z\in\mathbb{C}} |f(z)| e^{-\phi(z)} < \infty. \tag{1.2}$$

Moreover, we write $L^p(\Omega)$ for the space $L^p(\Omega, dA)$ where $\Omega \subset \mathbb{C}$, and we abbreviate $L^p(\mathbb{C}, dA)$ as L^p. A positive Borel measure μ on $\mathbb{C}$ is called doubling if there exists some constant $C > 1$ such that

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$$\mu(D(z,2r)) \leq C\mu(D(z,r)) \tag{1.3}$$

for all $z \in \mathbb{C}$ and $r > 0$, where $D(z,r)$ is the open disk in $\mathbb{C}$ with center z and radius r. The smallest $C > 1$ is called the doubling constant for μ. Hence, for each $z \in \mathbb{C}$, $\lim_{r\to\infty} \mu(D(z,r)) = \infty$. It is well known that μ has no point mass, i.e.,

$$\mu(\partial D(z,r)) = \mu(\{z\}) = 0 \quad \text{for every } z \in \mathbb{C} \text{ and } r > 0, \tag{1.4}$$

and is nonzero and locally finite. That is,

$$0 < \mu(D(z,r)) < \infty \quad \text{for every } z \in \mathbb{C} \text{ and } r > 0. \tag{1.5}$$

Note that since for each $z \in \mathbb{C}$, $\lim_{r\to\infty} \mu(D(z,r)) = \infty$, the function $r \mapsto \mu(D(z,r))$ is an increasing homeomorphism from $(0,\infty)$ to itself. Therefore, for every $z \in \mathbb{C}$, there is a unique positive radius $\rho(z)$ such that $\mu(z,\rho(z)) = 1$. For more information on doubling measures see [20]. Denote by $H(\mathbb{C})$ the space of holomorphic functions on $\mathbb{C}$. Then the doubling Fock space F^p_ϕ is defined by

$$F^p_\phi = L^p_\phi \cap H(\mathbb{C}) \tag{1.6}$$

where ϕ is a subharmonic function, not identically zero on $\mathbb{C}$, and $d\mu = \triangle\phi\, dA$ is a doubling measure. As shown in [16], ρ^{-2} is a regularization of $\Delta\phi$. Indeed, Theorem 14 in [16] states that when ϕ is subharmonic and $\Delta\phi\, dA$ is a doubling measure, there exists a subharmonic function $\psi \in \mathcal{C}^\infty(\mathbb{C})$ and $C > 0$ such that $|\psi - \phi| \leq C$, $\Delta\psi\, dA$ a doubling measure, and $\Delta\psi \simeq \rho_\psi^{-2} \simeq \rho_\phi^{-2}$. The comparability relation $\simeq$ is explained at the beginning of Section 2. Since the spaces of functions and sequences that we consider do not change if ϕ is replaced by ψ, we will assume that $\phi \in \mathcal{C}^\infty(\mathbb{C})$ and $\Delta\phi dA \simeq dA/\rho^2$ is a doubling measure. Hence, up to normalization by a constant, we can consider $\rho^{-2}(z) dz \otimes d\bar{z}$ to be the metric tensor describing the underlying geometry of our space.

It is well known that $(F^p_\phi, \|\cdot\|_{p,\phi})$ is a Banach space for $1 \leq p \leq \infty$ and a quasi-Banach space for $0 < p < 1$. Let $K_z = K(\cdot, z)$ be the reproducing kernel of F^2_ϕ. Then the orthogonal projection $P : L^2_\phi \to F^2_\phi$ is given by

$$Pf(z) = \int_{\mathbb{C}} f(w)\overline{K_z(w)} e^{-2\phi(w)} dA(w). \tag{1.7}$$

Then as shown in [18], for any $1 \leq p \leq \infty$, P is a bounded linear operator from L^p_ϕ to F^p_ϕ, and for any $f \in F^p_\phi$, $f = Pf$. Let $\Gamma = \text{span}\{K_z : z \in \mathbb{C}\}$, and consider the class of symbols

$$\mathcal{S} = \{f \text{ measurable} : fg \in L^2_\phi \text{ for } g \in \Gamma\}.$$

Note that $L^\infty \subset \mathcal{S}$. Given $f \in \mathcal{S}$, define the Toeplitz operator T_f and the Hankel operator H_f on F^p_ϕ by

$$T_f g = P(fg), \quad H_f g = (I - P)(fg) = fg - P(fg). \tag{1.8}$$

The doubling Fock spaces as well as some pointwise estimates of the Bergman kernel have been studied in seminal papers of Christ [3], and Marco, Massaneda and Ortega-Ceda [16,17]. Oliver and Pascuas [18] studied the characterization of boundedness, compactness and the Schatten class membership of Toeplitz operators on doubling Fock spaces. In [11], Hu and Virtanen introduced a new space IDA of locally integrable functions whose integral distance to holomorphic functions is finite and used it to characterize boundedness and compactness of Hankel operators on weighted Fock spaces. Using the same notion, in [9] they characterized

Schatten class Hankel operators acting on weighted Fock spaces F^2_Φ, where $m \le \triangle\Phi \le M$ for some $m, M > 0$. Recently, their characterizations of bounded and compact Hankel operators was extended to the setting of doubling Fock spaces in [15].

In the present work, we use a generalized version of IDA to study the Schatten class membership of Hankel operators on doubling Fock spaces. Of particular interest is the result of Berger and Coburn [2] which says that, for $f \in L^\infty$, if H_f is a compact operator acting on the classical Fock space F^2, then so is $H_{\bar{f}}$. We refer to this property as the Berger-Coburn phenomenon and note that an analogous statement fails both in the Hardy and Bergman spaces (see, e.g., [6]). More recently, Berger and Coburn's result has been extended to Fock spaces with standard weights by Hagger and Virtanen [6] (using limit operator techniques as opposed to C^*-algebra techniques and Hilbert space methods) and to generalized Fock spaces F^p_Φ by Hu and Virtanen [11]. Our approach is similar to that of [11] except that we need to deal with more complicated geometry induced by the function ρ arising in the study of doubling Fock spaces.

It is natural to ask whether the Berger-Coburn phenomenon also holds for Schatten class Hankel operators. Indeed, Bauer [1] was the first to show that this property holds for Hilbert-Schmidt Hankel operators on F^2. Recently, Hu and Virtanen in [9] proved that when $1 < p < \infty$, H_f acting on F^2_Φ is in the Schatten class S_p if and only if $H_{\bar{f}}$ is in S_p. This was followed by the work of Xia [21], in which he showed also that if $f(z) = 1/z$ for $|z| > 1$ and $f = 0$ elsewhere, then H_f acting on the classical Fock space F^2 is in the trace class while $H_{\bar{f}}$ is not. In his work, Xia employed a rather long and involved calculations using the standard basis vectors $e_k(z) = z^k/\sqrt{k!}$ and the reproducing kernel $K(z, w) = e^{z\bar{w}}$. Observe that for non-standard weighted Fock spaces, there are no explicit formulas for the basis vectors or the reproducing kernel. To overcome this, Hu and Virtanen [12] used their characterizations of Schatten class Hankel operators to verify that Xia's example shows that the Berger-Coburn phenomenon fails for $S_p(F^2_\varphi, L^2_\varphi)$ when $0 < m < \Delta\varphi < M$ and $0 < p \le 1$. Here, we use an analogous approach on doubling Fock spaces to prove the existence of the Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators. When $0 < p \le 1$, we show that the Berger-Coburn phenomenon fails for some doubling Fock spaces—the larger the value of p, the fewer Fock spaces we can cover.

To state our main results, following [11,14] with a modification according to the doubling property of the measure under consideration, we define

$$(G_{q,r}(f)(z))^q = \inf_{h\in H(D^r(z))} \frac{1}{|D^r(z)|} \int\limits_{D^r(z)} |f-h|^q dA \tag{1.9}$$

for $f \in L^q_{loc}$, $q \ge 1$ and $r > 0$. Here $|D^r(z)|$ is the Lebesgue measure of $D^r(z) := D(z, r\rho(z))$. Now, for $0 < p \le \infty$, $1 \le q \le \infty$, and $\alpha \in \mathbb{R}$, the space $\mathrm{IDA}^{p,q,\alpha}_r$ consists of all $f \in L^q_{loc}$ such that $\|f\|_{\mathrm{IDA}^{p,q,\alpha}_r} = \|\rho^\alpha G_{q,r}(f)\|_{L^p} < \infty$. Besides, for $f \in L^1_{loc}$, define $\hat{f}_r(z) := |D^r(z)|^{-1} \int_{D^r(z)} f dA$.

Theorem 1.1 *(IDA decomposition). Let $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic such that $d\mu = \Delta\phi dA$ is a doubling measure. Suppose that $1 \le q \le \infty$, $0 < p < \infty$, $\alpha \in \mathbb{R}$, and $f \in L^q_{loc}$. Then for $f \in \mathrm{IDA}^{p,q,\alpha}_r$, $f = f_1 + f_2$ where $f_1 \in \mathcal{C}^2(\mathbb{C})$ and*

$$\rho^{1+\alpha}|\bar{\partial} f_1| + \rho^{1+\alpha}(\widehat{|\bar{\partial} f_1|^q}_r)^{1/q} + \rho^\alpha(\widehat{|f_2|^q}_r)^{1/q} \in L^p, \tag{1.10}$$

for some (equivalent any) $r > 0$, and

$$\|f\|_{\mathrm{IDA}^{p,q,\alpha}_r} \simeq \inf\left\{ \|\rho^{1+\alpha}(\widehat{|\bar{\partial} f_1|^q}_r)^{1/q}\|_{L^p} + \|\rho^\alpha(\widehat{|f_2|^q}_r)^{1/q}\|_{L^p} \right\}, \tag{1.11}$$

where the infimum is taken over all possible decompositions $f = f_1 + f_2$, with f_1 and f_2 satisfying the conditions in (3.11).

Theorem 1.1 was stated in [15] without proof. We believe that the proof is rather technical and not trivial at all. It appears that this theorem should be a natural extension of Theorem 3.8 in [11]. However, bounding a solution to the $\bar{\partial}$-equation in the doubling Fock space is problematic.

Theorem 1.2 *(Schatten class membership of Hankel operators). Let $0 < p \leq \infty$, and $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic such that $d\mu := \Delta\phi dA$ is a doubling measure. Then for $f \in \mathcal{S}$, the following are equivalent:*

(1) *$H_f : F^2_\phi \to L^2_\phi$ is in S_p,*
(2) *$f \in \mathrm{IDA}^{p,2,-2/p}_r$, for some (equivalent any) $r > 0$.*

Moreover,

$$\|H_f\|_{S_p} \simeq \|f\|_{\mathrm{IDA}^{p,2,-2/p}_r}. \tag{1.12}$$

Remark. Assuming smoothness of ρ^{-2}, the condition for the S_p membership of the Hankel operator on the doubling Fock space is equivalent to the condition that $G_{2,r}(f)$ belongs to the space of L^p functions on $\mathbb{C}$ with the conformal metric $\rho^{-2}dz \otimes d\bar{z}$.

To characterize the simultaneous membership of H_f and $H_{\bar{f}}$ in S_p, we need to define the space of integral mean oscillation. First, for $f \in L^2_{loc}$ and $r > 0$, the mean oscillation of f is defined by

$$MO_{2,r}(f)(z) = \left(\frac{1}{|D^r(z)|} \int\limits_{D^r(z)} |f - \hat{f}_r(z)|^2 dA \right)^{1/2}. \tag{1.13}$$

Given $0 < p \leq \infty$ and $\alpha \in \mathbb{R}$, we define the space $\mathrm{IMO}^{p,2,\alpha}_r$ to be the family of those $f \in L^2_{loc}$ such that

$$\|f\|_{\mathrm{IMO}^{p,2,\alpha}_r} = \|\rho^\alpha MO_{2,r}(f)\|_{L^p} < \infty. \tag{1.14}$$

Theorem 1.3. *Let $0 < p < \infty$ and assume that $\phi \in \mathcal{C}^\infty(\mathbb{C})$ is subharmonic such that $d\mu = \Delta\phi dA$ is a doubling measure. Then the following are equivalent.*

(1) *Both H_f and $H_{\bar{f}} \in S_p(F^2_\phi, L^2_\phi)$,*
(2) *$f \in \mathrm{IMO}^{p,2,-2/p}_r$, for some (equivalent any) $r > 0$. Moreover,*

$$\|H_f\|_{S_p} + \|H_{\bar{f}}\|_{S_p} \simeq \|f\|_{\mathrm{IMO}^{p,2,-2/p}_r}. \tag{1.15}$$

Using the preceding result, it is easy to show that $H_{\bar{f}}$ is not Hilbert-Schmidt on F^2_ϕ when f is a non-constant entire function (see Theorem 5.4), which implies an analogous result of Schneider [19] for the canonical weights $\phi(z) = |z|^m$ and $f(z) = z^k$ when k is a positive integer and $m > 0$. However, when we restrict our study to bounded symbols, it turns out that $H_{\bar{f}} \in S_2$ whenever $H_f \in S_2$ as seen in the following theorem.

Theorem 1.4 *(Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators). Let $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic and suppose that $d\mu = \Delta\phi\, dA$ is a doubling measure. Then for $f \in L^\infty$, $H_f \in S_2(F^2_\phi, L^2_\phi)$ if and only if $H_{\bar{f}} \in S_2(F^2_\phi, L^2_\phi)$, with*

$$\|H_{\bar{f}}\|_{S_2} \simeq \|H_f\|_{S_2}. \tag{1.16}$$

It is worth emphasizing that the preceding theorem for Hilbert-Schmidt Hankel operators was proved by Bauer [1] in 2004, and it took almost two decades until it was proved for other Schatten classes by Hu and Virtanen [9]. This leads to the following question.

Open Problem 1.5. Does the Berger-Coburn phenomenon hold true for other Schatten classes S_p when $1 < p < \infty$?

For a discussion on the preceding open problem (involving the Muckenhoupt condition for the boundedness of the Beurling-Ahlfors operator), see Remark 6.1 in Section 6.

Before stating our last theorem, we recall the following growth condition for the function ρ. Given a doubling Fock space F^2_ϕ, there are constants $C, \eta > 0$ and $0 \le \beta < 1$ such that

$$C^{-1}|z|^{-\eta} \le \rho(z) \le C|z|^{\beta} \tag{1.17}$$

for $|z| > 1$ (see Equation (5) of [16]); we denote the smallest β that satisfies (1.17) by β_ϕ.

The following result shows the Berger-Coburn phenomenon fails for $S_p(F^2_\phi, L^2_\phi)$ provided that β_ϕ is sufficiently small in comparison with the value of p.

Theorem 1.6. *Let $\phi \in \mathcal{C}^\infty(\mathbb{C})$ be subharmonic with $d\mu = \Delta\phi\, dA$ a doubling measure. Then, for $0 < p \le 1$ with $\beta_\phi \le \frac{1-p}{1-p/2}$, the Berger-Coburn phenomenon for Schatten class Hankel operators fails; that is, there is an $f \in L^\infty(\mathbb{C})$ such that $H_f \in S_p(F^2_\phi, L^2_\phi)$ but $H_{\bar{f}} \notin S_p(F^2_\phi, L^2_\phi)$.*

In particular, when ρ is bounded, the Berger-Coburn phenomenon fails for all $0 < p \le 1$.

A simple consequence of the preceding theorem is that if F^2_ϕ is a doubling Fock space, then the Berger-Coburn phenomenon fails for $S_p(F^2_\phi, L^2_\phi)$ provided that p is sufficiently small.

Another consequence is the following corollary, in which we consider again the canonical doubling weights $\phi(z) = |z|^m$ and determine when the Berger-Coburn phenomenon fails for these weights.

Corollary 1.7. *Let $m > 0$ and $0 < p \le 1$. Then the Berger-Coburn phenomenon fails for $S_p(F^2_{|z|^m}, L^2_{|z|^m})$ if*

$$m \ge \frac{p}{1 - \frac{p}{2}}.$$

In particular, if $m \ge 2$, then the phenomenon fails for all Schatten classes S_p with $0 < p \le 1$.

Theorem 1.6 and its corollary lead to the following question.

Open Problem 1.8. Determine whether the Berger-Coburn phenomenon fails for $S_p(F^2_\phi, L^2_\phi)$ when $0 < p \le 1$ and $\Delta\phi\, dA$ is doubling.

The paper is organized as follows. In the next section, we provide preliminaries on the reproducing kernel, including global and local estimates, and elaborate more on the radius function ρ and the induced metric on the complex plane. In Section 3, we provide useful lemmas and use them to prove Theorem 1.1 (IDA decomposition). In Section 4, we use Toeplitz operators with locally finite positive Borel measures to prove Theorem 1.2, which characterizes the Schatten class membership of Hankel operators. Section 5 is devoted to the study of the function space IMO of integral mean oscillation, which we use to prove Theorem 1.3. Finally, in Section 6, we prove the Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators on general doubling Fock spaces as stated in Theorem 1.4. We finish the last section with the proofs of Theorem 1.6 and Corollary 1.7.

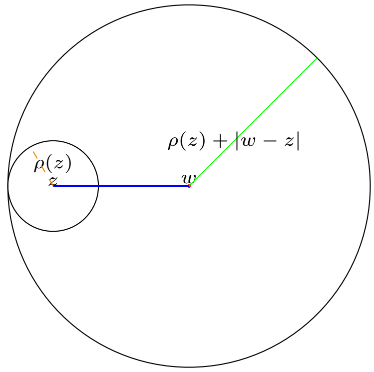

Fig. 1. Relation between $\rho(z)$ and $\rho(w)$.

2. Preliminaries

In this section we recall and prove some key lemmas on the function ρ, the reproducing kernel of F^2_ϕ, the space $\mathrm{IDA}^{p,q,\alpha}_r$, and their related integral and norm estimates.

Notation. We use C to denote positive constants whose value may change from line to line but does not depend on the functions being considered. We say that $A \simeq B$ if there exists a constant $C > 0$ such that $C^{-1}A \leq B \leq CA$. Moreover, $A \lesssim B$ if $A \leq CB$ for some positive constant C.

Let ϕ be a subharmonic function on $\mathbb{C}$ such that $d\mu = \Delta\phi dA$ is a doubling measure. Recall that there is a function ρ such that $\mu(D(z, \rho(z))) = 1$, for every point $z \in \mathbb{C}$. In other words, the radius of a disk with unit measure depends on the center of the disk. As shown in the Fig. 1, $D(z, \rho(z)) \subset D(w, |w - z| + \rho(z))$. Hence, $1 \leq \mu(D(w, |w - z| + \rho(z)))$, and thus $\rho(w) \leq \rho(z) + |w - z|$. By symmetry,

$$|\rho(w) - \rho(z)| \leq |w - z|, \quad \text{for every } z, w \in \mathbb{C}. \tag{2.1}$$

Lemma 2.1 *(See [18], Lemma 2.2). For every $r > 0$ there is a constant $c_r \geq 1$, depending only on r and the doubling constant for μ, such that*

$$c_r^{-1}\rho(z) \leq \rho(w) \leq c_r\rho(z), \quad \text{for every } z \in \mathbb{C} \text{ and } w \in D^r(z). \tag{2.2}$$

Namely, $c_r = (1 - r)^{-1}$, for every $0 < r < 1$. In other words, $\rho(w)$ and $\rho(z)$ are equivalent on a disk.

Consider the distance d_ϕ induced by the metric $\rho^{-2}dz \otimes d\bar{z}$. Indeed, for any $z, w \in \mathbb{C}$,

$$d_\phi(z, w) = \inf_\gamma \int_0^1 \frac{|\gamma'(t)|}{\rho(\gamma(t))} dt, \tag{2.3}$$

where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = w$.

Lemma 2.2 *(See [16], Lemma 4). There exists $\delta > 0$ such that for every $r > 0$ there exists $C_r > 0$ such that*

$$C_r^{-1}\frac{|z - w|}{\rho(z)} \leq d_\phi(z, w) \leq C_r\frac{|z - w|}{\rho(z)}, \quad \text{for } w \in D^r(z), \tag{2.4}$$

and

$$C_r^{-1}\Big(\frac{|z - w|}{\rho(z)}\Big)^\delta \leq d_\phi(z, w) \leq C_r\Big(\frac{|z - w|}{\rho(z)}\Big)^{2-\delta}, \quad \text{for } w \in \mathbb{C} \setminus D^r(z), \tag{2.5}$$

Now we can state the following pointwise estimate for the Bergman kernel.

Lemma 2.3.

(1) *There exist* $C, \epsilon > 0$ *such that*

$$|K(w,z)| \leq C \frac{e^{\phi(w)+\phi(z)}}{\rho(w)\rho(z)} e^{-\left(\frac{|z-w|}{\rho(z)}\right)^{\epsilon}}, \quad w, z \in \mathbb{C}, \tag{2.6}$$

(2) *There exists some* $r_0 > 0$ *such that for* $z \in \mathbb{C}$ *and* $w \in D^{r_0}(z)$*, we have*

$$|K(w,z)| \simeq \frac{e^{\phi(w)+\phi(z)}}{\rho(z)^2}. \tag{2.7}$$

(3) $k_{p,z} \to 0$ *uniformly on compact subsets of* $\mathbb{C}$ *as* $z \to \infty$*, where* $k_{p,z} := \frac{K_z}{\|K_z\|_{p,\phi}}$ *is the normalized Bergman kernel of* F^p_ϕ.

(4) *For any* $1 \leq p \leq \infty$*, we have that*

$$\|K_z\|_{p,\phi} \simeq e^{\phi(z)} \rho(z)^{2/p-2}. \tag{2.8}$$

Proof. See Theorem 1.1 and Proposition 2.11 of [17] respectively for parts (1) and (2), Lemma 2.3 of [8] for part (3), and Proposition 2.9 of [18] for part (4). □

Given a sequence $\{a_j\}_{j=1}^{\infty} \subset \mathbb{C}$, and $r > 0$, we call $\{a_j\}_{j=1}^{\infty}$ an r-lattice if $\{D^r(a_j)\}_{j=1}^{\infty}$ covers $\mathbb{C}$ and the disks of $\{D^{r/5}(a_j)\}_{j=1}^{\infty}$ are pairwise disjoint. Moreover, for an r-lattice $\{a_j\}_{j=1}^{\infty}$, and a real number $m > 1$, there exists an integer N such that

$$1 \leq \sum_{j=1}^{\infty} \chi_{D^{mr}(a_j)(z)} \leq N \tag{2.9}$$

where χ_E is the characteristic function of a subset E of $\mathbb{C}$. For $f, e \in L^2_\phi$, the tensor product $f \otimes e$ as a rank one operator on L^2_ϕ is defined by

$$f \otimes e(g) = \langle g, e \rangle f, \quad g \in L^2_\phi. \tag{2.10}$$

Lemma 2.4. *Given* $r > 0$*, there is some constant* $C > 0$ *such that if* Γ *is an* r*-lattice in* $\mathbb{C}$*, and if* $\{e_a : a \in \Gamma\}$ *is an orthonormal set in* L^2_ϕ*, then*

$$\left\| \sum_{a \in \Gamma} k_{2,a} \otimes e_a \right\|_{L^2_\phi \to L^2_\phi} \leq C, \tag{2.11}$$

where $k_{2,a} := \frac{K_a}{\|K_a\|_{2,\phi}}$ *is the normalized Bergman kernel.*

Proof. Note that $\{\lambda_a = \langle g, e_a \rangle_{2,\phi}\}_{a \in \Gamma} \in l^2$. Then similar to the proof of Lemma 2.4 in [7],

$$\left\| \sum_{a \in \Gamma} \lambda_a k_{2,a} \right\| \leq C \left\| \{\lambda_a\}_{a \in \Gamma} \right\|_{l^2}, \tag{2.12}$$

where the constant C only depends on r. Then similar to the proof of Lemma 2.4 in [9], we have

$$\left\| \Big(\sum_{a\in\Gamma} k_{2,a} \otimes e_a \Big)(g) \right\|^2 \leq C|\langle g, e_a\rangle|^2 \leq C\|g\|^2. \quad \square \tag{2.13}$$

We finish this section with a description of ρ for the canonical weights $|z|^m$ with $m > 0$.

Lemma 2.5. *Let $\phi(z) = |z|^m$ with $m > 0$. Then $d\mu = \Delta\phi dA$ is a doubling measure. Moreover, there is an $R > 0$ such that*

$$\rho(z) \simeq |z|^{1-m/2}$$

for $|z| > R$. In particular, when $m \geq 2$, ρ is bounded.

Proof. Note that $\Delta\phi(z) = m^2|z|^{m-2}$. To show that $d\mu$ is a doubling weight, it is enough to prove that for any $x \geq 0$ and $r > 0$,

$$\int_{D(x,2r)} |z|^{m-2} dA(z) \leq C \int_{D(x,r)} |z|^{m-2} dA(z), \tag{2.14}$$

where the constant C is independent of x and r.

We consider $r > \frac{x}{100} \geq 0$ first. Then $D(x, 2r) \subset D(0, x + 2r)$, so that

$$\int_{D(x,2r)} d\mu(\xi) \leq \int_{|\xi|\leq x+2r} |\xi|^{m-2} dA(\xi) \leq \int_{|\xi|\leq 102r} |\xi|^{m-2} dA(\xi) \leq C_1 r^m. \tag{2.15}$$

On the other hand, if $m \geq 2$,

$$\int_{D(x,r)} d\mu(\xi) \geq \int_{D(x,r)\cap\{\mathrm{Re}\,\xi\geq x\}} d\mu(\xi) \geq \int_{D(0,r)\cap\{\mathrm{Re}\,\xi\geq 0\}} d\mu(\xi) \geq C_2 r^m. \tag{2.16}$$

From (2.15) and (2.16) we obtain (2.14) for $m \geq 2$ and $r > \frac{x}{100}$.

Now we suppose $0 < r < \frac{x}{100}$. Then

$$D(x, 2r) \subset \{te^{i\theta} : x - 2r < t < x + 2r, |\theta| < \arcsin \frac{2r}{x}\},$$
$$D(x, r) \supset \{te^{i\theta} : x - c_1 r < t < x + c_2 r, |\theta| < \arcsin \frac{r}{2x}\},$$

where c_1 and c_2 are positive constants independent of x are r. Hence,

$$\begin{aligned}\int_{D(x,2r)} d\mu &\leq \int_{x-2r}^{x+2r} r^{m-1} dr \int_{-\arcsin\frac{2r}{x}}^{\arcsin\frac{2r}{x}} d\theta \simeq \frac{r}{x}\left[(x + 2r)^m - (x - 2r)^m\right] \\ &\simeq \frac{r}{x} r x^{m-1} = r^2 x^{m-2},\end{aligned} \tag{2.17}$$

where the constants in the inequalities $\simeq$ are all independent of x and r. Similarly,

$$\int_{D(x,r)} d\mu \geq \int_{x-c_1 r}^{x+c_2 r} r^{m-1} dr \int_{-\arcsin\frac{r}{2x}}^{\arcsin\frac{r}{2x}} d\theta \tag{2.18}$$

$$\simeq \frac{r}{x}\left[(x+c_2r)^m - (x-c_1r)^m\right] \simeq r^2x^{m-2}.$$

Using (2.17) and (2.18), we obtain (2.14).

For $0 < m < 2$, and $r > \frac{x}{100}$,

$$\int\limits_{D(x,r)} |\xi|^{m-2}dA(\xi) = \int\limits_{D(0,r)} |\xi + x|^{m-2}dA(\xi) \geq \int\limits_{D(0,r)} |\xi|^{m-2}dA(\xi) \geq C_3r^m. \tag{2.19}$$

From (2.15) and (2.19) we obtain (2.14) for $0 < m < 2$ and $r > \frac{x}{100}$.

Now notice that using (2.17) and (2.18) and when x is large enough,

$$\int\limits_{D(x,x^{-\frac{m-2}{2}})} |\xi|^{m-2}dA(\xi) \simeq 1. \tag{2.20}$$

This, together with the doubling property implies that there exists $R > 0$ large enough, such that for the Fock space $F^2_{|z|^m}$,

$$\rho(z) \simeq |z|^{-\frac{m-2}{2}} = |z|^{1-\frac{m}{2}} \tag{2.21}$$

for $|z| \geq R$. □

3. The space IDA

The goal of this section is to prove the IDA decomposition Theorem 1.1. Before proving the theorem, we need to see some definitions and lemmas.

Lemma 3.1. *Suppose $1 \leq q < \infty$. Then for $f \in L^q_{loc}$, $z \in \mathbb{C}$, and $r > 0$, there is $h \in H(D^r(z))$ such that*

$$\left(\widehat{|f-h|^q}_r(z)\right)^{1/q} = G_{q,r}(f)(z), \tag{3.1}$$

and for $s < r$,

$$\sup_{w \in D^s(z)} |h(w)| \leq C\|f\|_{L^q(D^r(z),dA)}, \tag{3.2}$$

where the constant C is independent of f and r.

Proof. This proof is similar to the proof of Lemma 3.3 in [11]. Taking $h = 0$,

$$G_{q,r}(f)(z) \leq \left(\widehat{|f|^q}_r(z)\right)^{1/q} < \infty. \tag{3.3}$$

Then for $j = 1, 2, \cdots$, pick $h_j \in H(D^r(z))$ such that

$$\left(\widehat{|f-h_j|^q}_r(z)\right)^{1/q} \to G_{q,r}(f)(z) \quad \text{as } j \to \infty. \tag{3.4}$$

Hence for sufficiently large j,

$$\left(\widehat{|h_j|^q}_r(z)\right)^{1/q} \leq C\left\{\left(\widehat{|f-h_j|^q}_r(z)\right)^{1/q} + \left(\widehat{|f|^q}_r(z)\right)^{1/q}\right\} \leq C\left(\widehat{|f|^q}_r(z)\right)^{1/q}. \tag{3.5}$$

Thus, we can find a subsequence $\{h_{j_k}\}_{k=1}^{\infty}$ and a function $h \in H(D^r(z))$ such that $\lim_{k\to\infty} h_{j_k}(w) = h(w)$ for $w \in D^r(z)$. By (3.4),

$$G_{q,r}(f)(z) \le \left(\widehat{|f-h|^q}_r(z)\right)^{1/q} \le \liminf_{k\to\infty} \left(\widehat{|f-h_{j_k}|^q}_r(z)\right)^{1/q} = G_{q,r}(f)(z) \tag{3.6}$$

where in the RHS inequality we have used Fatou's Lemma. This gives us (3.1).

Now for $w \in D^s(z)$, by the mean value Theorem,

$$|h(w)| \le \left(\widehat{|h|^q}_s(z)\right)^{1/q} \le C\left(\widehat{|h|^q}_r(z)\right)^{1/q} \le \left(\widehat{|f|^q}_r(z)\right)^{1/q} = C\|f\|_{L^q(D^r(z),dA)}. \quad \square \tag{3.7}$$

Now we are ready to define f_1 and f_2 in Theorem 1.1. Using (2.2) and the triangle inequality, there exists $m \in (0,1)$ such that $D^{mr}(w) \subset D^r(z)$, whenever $w \in D^{mr}(z)$. For $r > 0$, let $\{a_j\}_{j=1}^{\infty}$ be a mr-lattice, and let $J_z := \{j : z \in D^r(a_j)\}$, so that $|J_z| = \sum_{j=1}^{\infty} \chi_{D^r(a_j)}(z) \le N$, for some integer N. Let $\eta : \mathbb{C} \to [0,1]$ be the following smooth function with bounded derivatives.

$$\eta(z) = \begin{cases} 1 & \text{if } |z| \le 1/2, \\ 0 & \text{if } |z| \ge 1. \end{cases} \tag{3.8}$$

For each $j \ge 1$ we define $\eta_j(z) = \eta(\frac{z-a_j}{mr\rho(a_j)})$. We can normalize η_j such that $\int_{\mathbb{C}} \eta_j dA = 1$, for each $j \ge 1$. Define $\psi_j(z) = \frac{\eta_j(z)}{\sum_{k=1}^{\infty} \eta_k(z)}$. Then one can see that $\{\psi_j\}_{j=1}^{\infty}$ is a partition of unity subordinate to $\{D^{mr}(a_j)\}_{j\ge1}$, satisfying the following properties:

$$\begin{aligned} \operatorname{Supp} \psi_j \subset D^{mr}(a_j), \quad \psi_j(z) \ge 0, \quad \sum_{j=1}^{\infty} \psi_j(z) = 1, \\ |\rho(a_j)\bar\partial\psi_j| \le C, \quad \sum_{j=1}^{\infty} \bar\partial\psi_j(z) = 0, \end{aligned} \tag{3.9}$$

where the constant C may depend on r.

By Lemma 3.1, for $j = 1, 2, \cdots$, we can pick $h_j \in H(D^r(a_j))$ such that

$$\widehat{|f-h_j|^q}_r(a_j) = \frac{1}{|D^r(a_j)|} \int_{D^r(a_j)} |f-h_j|^q dA = G_{q,r}(f)(a_j)^q. \tag{3.10}$$

For $1 \le q < \infty$ and $f \in L^q_{loc}$, decompose $f = f_1 + f_2$ as

$$f_1(z) := \sum_{j=1}^{\infty} h_j(z)\psi_j(z), \quad f_2(z) := f(z) - f_1(z). \tag{3.11}$$

Lemma 3.2. *Let $1 \le q < \infty$, $f \in L^q_{loc}$, and $r > 0$. Decomposing $f = f_1 + f_2$ as in (3.11), we have $f_1 \in \mathcal{C}^2(\mathbb{C})$ and*

$$\rho(z)|\bar\partial f_1(z)| + \rho(z)(\widehat{|\bar\partial f_1|^q}_{mr})^{1/q} + (\widehat{|f_2|^q}_{mr})^{1/q} \le C G_{q,R}(f)(z), \tag{3.12}$$

for some $R > r$ and $m \in (0,1)$.

Proof. Using the properties of h_j and ψ_j we can easily see that $f_1 \in \mathcal{C}^2(\mathbb{C})$. Let $z \in \mathbb{C}$, and $J_z = \{j : z \in D^r(a_j)\}$. We know that if $z \in D^r(a_j)$, then $\rho(z) \le C\rho(a_j)$. Therefore, knowing $\sum_{j=1}^{\infty} \bar\partial\psi_j = 0$, using (3.9), the triangle inequality, and since $|h_j - h_1|^q$ is plurisubharmonic on $D^r(a_j)$,

$$\begin{aligned}\rho(z)|\bar{\partial} f_1(z)| &= \rho(z)\left|\bar{\partial}(\sum_{j=1}^{\infty} h_j(z)\psi_j(z))\right| \le \rho(z)\sum_{j=1}^{\infty}|h_j(z)-h_1(z)||\bar{\partial}\psi_j(z)| \\ &\le C\sum_{j\in J_z}\left[\frac{1}{|D^r(a_j)|}\int_{D^r(a_j)}|h_j-h_1|^q dA\right]^{1/q}\rho(a_j)|\bar{\partial}\psi_j(z)| \\ &\le C\sum_{j\in J_z}\left[\frac{1}{|D^r(a_j)|}\int_{D^r(a_j)}\{|f-h_j|^q+|f-h_1|^q\}dA\right]^{1/q} \\ &\le C\sum_{j\in J_z}\left(\widehat{|f-h_j|^q}_r(a_j)\right)^{1/q}+\left(\widehat{|f-h_1|^q}_r(a_j)\right)^{1/q} \\ &\le C\sum_{j\in J_z} G_{q,r}(a_j) \le CG_{q,s}(f)(z), \end{aligned} \tag{3.13}$$

for some $s > r$, where the last inequality can be shown similarly to Corollary 3.4 in [11], and using the fact that $|J_z|$ is finite.

Moreover, note that

$$\begin{aligned}\rho(z)\left(\widehat{|\bar{\partial} f_1|^q}_{mr}(z)\right)^{1/q} &= \rho(z)\left[\frac{1}{|D^{mr}(z)|}\int_{D^{mr}(z)}|\bar{\partial} f_1(w)|^q dA(w)\right]^{1/q} \\ &\le C\left[\frac{1}{|D^{mr}(z)|}\int_{D^{mr}(z)}\rho(w)^q|\bar{\partial} f_1(w)|^q dA(w)\right]^{1/q} \\ &\le C\left[\frac{1}{|D^{mr}(z)|}\int_{D^{mr}(z)}G_{q,s}(f)(w)^q dA(w)\right]^{1/q} \\ &\le C\sup_{w\in D^{mr}(z)} G_{q,s}(f)(w) \le CG_{q,R}(f)(z), \end{aligned} \tag{3.14}$$

for some $R > s$, where again for the last inequality we use Corollary 3.4 in [11]. Similarly, since $\sum_{j=1}^{\infty}\psi_j = 1$,

$$|f_2(w)|^q = |f(w) - \sum_{j=1}^{\infty} h_j(w)\psi_j(w)|^q \le \sum_{j=1}^{\infty}|f(w)-h_j(w)|^q|\psi_j(w)|^q. \tag{3.15}$$

Hence, using $|\psi_j| \le 1$,

$$\begin{aligned}\left(\widehat{|f_2|^q}_{mr}(z)\right)^{1/q} &\le \sum_{j=1}^{\infty}\left[\frac{1}{|D^{mr}(z)|}\int_{D^{mr}(z)}|f-h_j|^q|\psi_j|^q dA\right]^{1/q} \\ &\le C\sum_{j\in J_z} G_{q,r}(f)(a_j) \le CG_{q,R}(f)(z), \end{aligned} \tag{3.16}$$

similar to the previous part for $\rho|\bar{\partial} f_1|$. Putting everything together, we can find a big enough $R > r$ such that (3.12) holds. □

Proof of Theorem 1.1. First, we show that if (1.10) holds for some r, then it holds for any r. Let $R > 0$. For $0 < r < R$ take $t = \frac{r}{2C_2R}$ and take $z_1, \cdots, z_N$ in the unit disk $D(0,1)$ so that $D(0,1) \subset \cup_{j=1}^N D(z_j, t)$. Set $a_j(z) = z + R\rho(z)z_j$. Then

$$D^R(z) \subset \cup_{j=1}^N D(z + R\rho(z)z_j, tR\rho(z)) \subset \cup_{j=1}^N D(a_j(z), \frac{r}{2}\rho(a_j(z))) = \cup_{j=1}^N D^{r/2}(a_j(z)). \tag{3.17}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{C}} \left(\widehat{|g|^q}_R(z)\right)^s dA(z) &\le C \int_{\mathbb{C}} \sum_{j=1}^N \left(\widehat{|g|^q}_{r/2}(a_j(z))\right)^s dA(z) \\ &\le C \int_{\mathbb{C}} dA(z) \sum_{j=1}^N \frac{1}{|D^{cr}(a_j(z))|} \int_{D^{cr}(a_j(z))} \left(\widehat{|g|^q}_r(u)\right)^s dA(u) \\ &= C \int_{\mathbb{C}} \left(\widehat{|g|^q}_r(u)\right)^s dA(u) \sum_{j=1}^N \int_{\mathbb{C}} \chi_{D^{cr}(a_j(z))}(u) \frac{1}{|D^{cr}(a_j(z))|} dA(z) \\ &\le C \int_{\mathbb{C}} \left(\widehat{|g|^q}_r(u)\right)^s dA(u),\end{aligned} \tag{3.18}$$

where for the second inequality take $c > 0$ such that $D^{cr}(a_j(z)) \subset \cap_{u \in D^{cr}(a_j(z))} D^r(u)$. Taking $s = p/q$ implies that (1.10) holds for some $r > 0$, if and only if it holds for any r.

Now assume that $f \in \mathrm{IDA}_r^{p,q,\alpha}$. That is, $f \in L^q_{loc}$ with $\|\rho^\alpha G_{q,r}(f)\|_{L^p} < \infty$. Decompose $f = f_1 + f_2$ as in Lemma 3.2. Then $f_1 \in \mathcal{C}^2(\mathbb{C})$, and (3.12) holds. Multiplying both sides with ρ^α and taking the L^p-norm, we obtain (1.10). □

4. Schatten class Hankel operators on doubling Fock spaces

Recall that for a bounded linear operator $T : H_1 \to H_2$ between two Hilbert spaces, the singular values λ_n are defined by

$$\lambda_n = \lambda_n(T) = \inf\{\|T - K\| : K : H_1 \to H_2, \operatorname{rank} K \le n\}. \tag{4.1}$$

The operator T is compact if and only if $\lambda_n \to 0$. Given $0 < p < \infty$, we say that T is in the Schatten class S_p and write $T \in S_p(H_1, H_2)$, if its singular value sequence $\{\lambda_n\}$ belongs to l^p. Then $\|T\|_{S_p}^p = \sum_{n=0}^\infty |\lambda_n|^p$ defines a norm when $1 \le p < \infty$ and a quasinorm when $0 < p < 1$. Moreover, for the quasi-Banach case, we have the triangle inequality.

$$\|T + S\|_{S_p}^p \le \|T\|_{S_p}^p + \|S\|_{S_p}^p, \quad \text{when } T, S \in S_p,\ 0 < p < 1, \tag{4.2}$$

which is called the Rotfel'd inequality. For a positive compact operators T on H and $p > 0$, $T \in S_p$ if and only if $T^p \in S_1$. Moreover, $\|T\|_{S_p}^p = \|T^p\|_{S_1}$. See [22] for further details on the properties of Schatten class operators, as well as the proof of the next two theorems.

Theorem 4.1 *(See [22], Theorem 1.26). If T is a compact operator on H and $p > 0$, then $T \in S_p$ if and only if $|T|^p = (T^*T)^{p/2} \in S_1$, if and only if $T^*T \in S_{p/2}$. Moreover,*

$$\|T\|_{S_p}^p = \||T|\|_{S_p}^p = \||T|^p\|_{S_1} = \|T^*T\|_{S_{p/2}}^{p/2}. \tag{4.3}$$

Consequently, $T \in S_p$ if and only if $|T| \in S_p$.

Theorem 4.2 *(See [22], Theorem 1.28). Suppose T is a compact operator on H and $p \geq 1$. Then T is in S_p if and only if*

$$\sum |\langle Te_n, \sigma_n\rangle|^p < \infty, \tag{4.4}$$

for all orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$. If T is positive, we also have

$$\|T\|_{S_p} = \sup\Big\{\big[\sum |\langle Te_n, \sigma_n\rangle|^p\big]^{1/p} : \quad \{e_n\} \text{ and } \{\sigma_n\} \text{ are orthonormal}\Big\}. \tag{4.5}$$

Given a locally finite positive Borel measure μ on $\mathbb{C}$, we define the Toeplitz operator T_μ with symbol μ as

$$T_\mu f(z) = \int_{\mathbb{C}} f(w)\overline{K_z(w)}e^{-2\phi(w)}d\mu(w). \tag{4.6}$$

Moreover, for every $r > 0$, the r-averaging transform of μ is defined by

$$\hat{\mu}_r(z) := \frac{\mu(D^r(z))}{|D^r(z)|} \simeq \frac{\mu(D^r(z))}{\rho(z)^2}. \tag{4.7}$$

Theorem 4.3 *(See [18], Theorem 4.1). Let μ be a locally finite positive Borel measure on $\mathbb{C}$, and let $0 < p < \infty$. Then the following are equivalent.*

(1) $T_\mu \in S_p(F_\phi^2)$,
(2) *There is $r_0 > 0$ such that any r-lattice $\{z_j\}_{j\geq 1}$ with $r \in (0, r_0)$ satisfies $\{\hat{\mu}_r(z_j)\}_{j\geq 1} \in l^p$,*
(3) *There is an r-lattice $\{z_j\}_{j\geq 1}$ such that $\{\hat{\mu}_r(z_j)\}_{j\geq 1} \in l^p$,*
(4) *There is $r > 0$ such that $\hat{\mu}_r \in L^p(\mathbb{C}, d\sigma)$,*

Moreover, $\|T_\mu\|_{S_p}^p \simeq \|\hat{\mu}_r\|_{L^p(\mathbb{C},d\sigma)}$, where $d\sigma = dA/\rho^2$.

The rest of this section is devoted to the proof of the Schatten class membership of the Hankel operators Theorem 1.2. For this purpose, let $a \in \mathbb{C}$ and $r > 0$. Let $A^2(D^r(a), e^{-2\phi}dA)$ be the weighted Bergman space containing the holomorphic functions in $L^2(D^r(a), e^{-2\phi}dA)$. Let $P_{a,r} : L^2(D^r(a), e^{-2\phi}dA) \to A^2(D^r(a), e^{-2\phi}dA)$ be the orthogonal projection, and for $f \in L^2(D^r(a), e^{-2\phi}dA)$, extend $P_{a,r}(f)$ to $\mathbb{C}$ by setting

$$P_{a,r}(f)_{|\mathbb{C}\setminus D^r(a)} = 0. \tag{4.8}$$

One can check that for $f, g \in L_\phi^2$,

$$P_{a,r}^2(f) = P_{a,r}(f), \quad \text{and} \quad \langle f, P_{a,r}(g)\rangle = \langle P_{a,r}(f), g\rangle. \tag{4.9}$$

Moreover, for $h \in F_\phi^2$,

$$P_{a,r}(h) = \chi_{D^r(a)}h, \quad \text{and} \quad \langle h, \chi_{D^r(a)}f - P_{a,r}(f)\rangle = 0. \tag{4.10}$$

Proof of Theorem 1.2. Here we borrow an idea from the proof of Proposition 6.8 in [5] and the proof of Theorem 1.1 in [9]. First we show that (2) $\implies$ (1). Let $f \in \mathrm{IDA}_r^{p,2,-2/p}$. Then by Theorem 1.1, $f = f_1 + f_2$ with

$$\rho^{1-2/p}|\bar{\partial} f_1| + \rho^{1-2/p}(\widehat{|\bar{\partial} f_1|^2}_r)^{1/2} + \rho^{-2/p}(\widehat{|f_2|^2}_r)^{1/2} \in L^p \tag{4.11}$$

Applying the definition,

$$\rho^{1-2/p}(z)(\widehat{|\bar{\partial} f_1|^2}_r(z))^{1/2} = \rho^{1-2/p}(z)\Big\{\frac{1}{|D^r(z)|}\int\limits_{D^r(z)} |\bar{\partial} f_1|^2 dA\Big\}^{1/2}, \tag{4.12}$$

and

$$\rho^{-2/p}(z)(\widehat{|f_2|^2}_r(z))^{1/2} = \rho^{-2/p}(z)\Big\{\frac{1}{|D^r(z)|}\int\limits_{D^r(z)} |f_2|^2 dA\Big\}^{1/2}. \tag{4.13}$$

Set $\Phi := \rho|\bar{\partial} f_1|$ or $\Phi = |f_2|$, and $\mu := |\Phi|^2$. First, if $\Phi = \rho|\bar{\partial} f_1|$,

$$\hat{\mu}_r(z) := \frac{\mu(D^r(z))}{|D^r(z)|} = \frac{1}{|D^r(z)|}\int\limits_{D^r(z)} |\Phi|^2 dA = \frac{1}{|D^r(z)|}\int\limits_{D^r(z)} \rho^2|\bar{\partial} f_1|^2 dA. \tag{4.14}$$

We claim that for $f \in \mathrm{IDA}_r^{p,2,-2/p}$, $\hat{\mu}_r \in L^{p/2}(\mathbb{C}, d\sigma)$. Note that

$$\begin{aligned}\|\hat{\mu}_r\|^{p/2}_{L^{p/2}(\mathbb{C},d\sigma)} &= \int\limits_{\mathbb{C}} |\hat{\mu}_r|^{p/2} dA/\rho^2 \\ &= \int\limits_{\mathbb{C}} \frac{1}{|D^r(z)|^{p/2}}\Big[\int\limits_{D^r(z)} \rho^2|\bar{\partial} f_1|^2 dA\Big]^{p/2} \frac{dA(z)}{\rho(z)^2}.\end{aligned} \tag{4.15}$$

Since $f \in \mathrm{IDA}_r^{p,2,-2/p}$, we have $\rho^{1-2/p}(\widehat{|\bar{\partial} f_1|^2}_r)^{1/2} \in L^p$ and thus

$$\int\limits_{\mathbb{C}} \rho^{p-2}\Big\{\frac{1}{|D^r(z)|}\int\limits_{D^r(z)} |\bar{\partial} f_1|^2 dA\Big\}^{p/2} dA(z) < \infty. \tag{4.16}$$

Recall that in (4.15), $w \in D^r(z)$, and therefore there is a constant C such that $\rho(w) \le C\rho(z)$. Hence,

$$\|\hat{\mu}_r\|^{p/2}_{L^{p/2}(\mathbb{C},d\sigma)} \le \int\limits_{\mathbb{C}} \frac{C\rho(z)^{p-2}}{|D^r(z)|^{p/2}}\Big\{\int\limits_{D^r(z)} |\bar{\partial} f_1|^2 dA\Big\}^{p/2} dA(z) \asymp \text{LHS of (4.16)} < \infty. \tag{4.17}$$

Thus, we can conclude that $\hat{\mu}_r \in L^{p/2}(\mathbb{C}, d\sigma)$, for $\mu = \rho^2|\bar{\partial} f_1|^2$. Now, using Theorem 4.3, $T_\mu \in S_{p/2}(F^2_\phi)$.

Consider the multiplication $M_\Phi : F^2_\phi \to L^2_\phi$ defined by $M_\Phi f := \Phi f$. Then M_Φ is bounded for $\Phi = \rho|\bar{\partial} f_1|$ or $\Phi = |f_2|$. For $h, g \in L^2_\phi$,

$$\langle M^*_\Phi M_\Phi g, h\rangle_{2,\phi} = \langle M_\Phi g, M_\Phi h\rangle_{2,\phi} = \int\limits_{\mathbb{C}} g\bar{h}e^{-2\phi} dA = \langle T_{|\Phi|^2} g, h\rangle_{2,\phi}, \tag{4.18}$$

so, $M^*_\Phi M_\Phi = T_{|\Phi|^2} \in S_{p/2}$, and thus $M_\Phi \in S_p$. Moreover,

$$\|M_\Phi\|_{S_p} \simeq \|M_\Phi^* M_\Phi\|_{S_{p/2}} \simeq \|T_\mu\|_{S_{p/2}} \simeq \|\hat{\mu}_r\|_{L^{p/2}(\mathbb{C},d\sigma)}. \tag{4.19}$$

By equations (3.13) and (3.17) in [15], and using Fock-Carleson measures for F^2_ϕ, we can see that

$$\|H_{f_1}g\|_{2,\phi} \leq \|\rho g \bar{\partial} f_1\|_{2,\phi}, \quad \text{and} \quad \|H_{f_2}g\|_{2,\phi} \leq \|g f_2\|_{2,\phi}. \tag{4.20}$$

Therefore,

$$\|H_{f_1}\|_{S_p} \lesssim \|M_\Phi\|_{S_p} \simeq \|\hat{\mu}_r\|_{L^{p/2}(\mathbb{C},d\sigma)} \lesssim \|\rho^{1-2/p}(\widehat{|\bar{\partial} f_1|^2}_r)^{1/2}\|_{L^p} \asymp \|f\|_{\mathrm{IDA}^{p,2,-2/p}_r}. \tag{4.21}$$

To complete the proof, it remains to note that when $\mu = |f_2|^2$, we have

$$\begin{aligned}
\|\hat{\mu}_r\|^{p/2}_{L^{p/2}(\mathbb{C},d\sigma)} &= \int_{\mathbb{C}} \frac{1}{|D^r(z)|^{p/2}} \Big[\int_{D^r(z)} |f_2|^2 dA \Big]^{p/2} \frac{dA(z)}{\rho(z)^2} \\
&= \int_{\mathbb{C}} \Big[\frac{\rho(z)^{-2/p}}{|D^r(z)|^{1/2}} \Big\{ \int_{D^r(z)} |f_2|^2 dA \Big\}^{1/2} \Big]^p dA(z) \\
&= \|\rho^{-2/p}(\widehat{|f_2|^2}_r)^{1/2}\|_{L^p},
\end{aligned} \tag{4.22}$$

so that

$$\|H_{f_2}\|_{S_p} \lesssim \|f\|_{\mathrm{IDA}^{p,2,-2/p}_r}.$$

Consequently, $\|H_f\|_{S_p} \lesssim \|H_{f_1}\|_{S_p} + \|H_{f_2}\|_{S_p} \lesssim \|f\|_{\mathrm{IDA}^{p,2,-2/p}_r}$, and so $H_f \in S_p(F^2_\phi, L^2_\phi)$.

To show (1) $\implies$ (2) for $p \geq 1$, we proceed as follows. Recall that $\{a_j\}_{j=1}^\infty$ is an r-lattice if $\{D^r(a_j)\}_{j=1}^\infty$ covers $\mathbb{C}$ and $D^{r/5}(a_j) \cap D^{r/5}(a_k) = \emptyset$ for $j \neq k$. Let Γ be an r-lattice, and let $\{e_a : a \in \Gamma\}$ be an orthonormal basis of F^2_ϕ. Define linear operators T and B by

$$T = \sum_{a \in \Gamma} k_{2,a} \otimes e_a, \quad \text{and} \quad B = \sum_{a \in \Gamma} g_a \otimes e_a, \tag{4.23}$$

where

$$g_a = \begin{cases} \frac{\chi_{D^r(a)} H_f(k_{2,a})}{\|\chi_{D^r(a)} H_f(k_{2,a})\|} & \text{if } \|\chi_{D^r(a)} H_f(k_{2,a})\| \neq 0, \\ 0 & \text{if } \|\chi_{D^r(a)} H_f(k_{2,a})\| = 0. \end{cases} \tag{4.24}$$

Since $\|g_a\| \leq 1$ and $\langle g_a, g_b \rangle = 0$ when $a \neq b$, $\|B\|_{L^2_\phi} \to L^2_\phi \leq 1$. Moreover, by Lemma 2.4, we can see that $\|T\| \leq C$ for some constant C. Let $H_f \in S_p$. So in particular, H_f is compact. We know from Lemma 2.3 that $k_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $z \to \infty$, where $k_{p,z} = K_z/\|K_z\|_{p,\phi}$ is the normalized Bergman kernel for F^p_ϕ. By compactness of H_f we obtain that

$$\lim_{z \to \infty} \|\chi_{D^r(z)} H_f(k_{2,z})\|_{L^2_\phi} = 0. \tag{4.25}$$

Note that

$$\begin{aligned}
\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_a \rangle &= \langle \chi_{D^r(a)} H_f \sum_{b \in \Gamma} k_{2,b} \otimes e_b(e_a), \sum_{d \in \Gamma} g_d \otimes e_d(e_a) \rangle \\
&= \langle \chi_{D^r(a)} H_f(k_{2,a}), g_a \rangle = \|\chi_{D^r(z)} H_f(k_{2,z})\|_{L^2_\phi},
\end{aligned} \tag{4.26}$$

and

$$\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_b \rangle = 0, \quad a \neq b. \tag{4.27}$$

Thus, $B^* M_{\chi_{D^r(a)}} H_f T$ is a compact positive operator on L^2_ϕ. By Theorem 4.2, and since we are dealing with the case of $p \geq 1$,

$$\|B^* M_{\chi_{D^r(a)}} H_f T\|^p_{S_p} = \sup \left\{ \sum |\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_a \rangle : \{e_a\}_{a\in\Gamma} : \text{orthonormal} \right\}. \tag{4.28}$$

So,

$$\sum_{a\in\Gamma} |\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_a \rangle| \leq \|B^* M_{\chi_{D^r(a)}} H_f T\|^p_{S_p} \leq C\|H_f\|^p_{S_p}, \tag{4.29}$$

as $\|B\| \leq 1$, $\|M_{\chi_{D^r(a)}}\| \leq 1$, and $\|T\| \leq C$. Recall that

$$G_{2,r}(f)(a) = \inf \left\{ \left(\frac{1}{|D^r(a)|} \int\limits_{D^r(a)} |f-h|^2 dA \right)^{1/2} : h \in H(D^r(a)) \right\}, \tag{4.30}$$

and for $1 \leq p < \infty$, $\|K_z\|_{p,\phi} \asymp e^{\phi(z)} \rho(z)^{2/p-2}$. Moreover, recalling Lemma 2.3 there exists $r_0 > 0$ such that for $w \in D^{r_0}(z)$,

$$|K(w,z)| \asymp \frac{e^{\phi(w)+\phi(z)}}{\rho(z)^2}. \tag{4.31}$$

Thus for $w \in D^{r_0}(z)$,

$$|k_{p,z}(w)|e^{-\phi(w)} = \frac{|K(w,z)|}{\|K_z\|_{p,\phi}} e^{-\phi(w)} \asymp \frac{e^{\phi(w)+\phi(z)} e^{-\phi(w)}}{\rho(z)^2 e^{\phi(z)}} \rho(z)^{-2/p+2} = \rho(z)^{-2/p} > 0, \tag{4.32}$$

and we can conclude that $\frac{P(fk_{2,z})}{k_{2,z}} \in H(D^r(z))$. Hence,

$$G_{2,r}(f)(a) \leq \left[\frac{1}{|D^r(a)|} \int\limits_{D^r(a)} |f - \frac{P(fk_{2,a})}{k_{2,a}}|^2 dA \right]^{1/2}. \tag{4.33}$$

Moreover,

$$\begin{aligned}
\|\chi_{D^r(a)} H_f(k_{2,a})\|_{L^2_\phi} &= \left[\int\limits_{D^r(a)} |fk_{2,a} - P(fk_{2,a})|^2 e^{-2\phi} dA \right]^{1/2} \\
&= \left[\int\limits_{D^r(a)} |f - \frac{P(fk_{2,a})}{k_{2,a}}|^2 |k_{2,a}|^2 e^{-2\phi} dA \right]^{1/2} \\
&\overset{(4.32)}{\asymp} \left[\int\limits_{D^r(a)} |f - \frac{P(fk_{2,a})}{k_{2,a}}|^2 \rho(a)^{-2} dA \right]^{1/2}
\end{aligned}$$

$$\asymp \left[\frac{1}{|D^r(a)|} \int\limits_{D^r(a)} |f - \frac{P(fk_{2,a})}{k_{2,a}}|^2 dA \right]^{1/2}, \tag{4.34}$$

where in the last line we have used the equivalence $|D^r(z)| \asymp \rho(z)^2$. Hence,

$$G_{2,r}(f)(a) \lesssim \|\chi_{D^r(a)} H_f(k_{2,a})\|_{L^2_\phi}, \tag{4.35}$$

and therefore,

$$\begin{aligned} \sum_{a\in\Gamma} G_{2,r}(f)(a)^p &\lesssim \sum_{a\in\Gamma} \|\chi_{D^r(a)} H_f(k_{2,a})\|^p_{L^2_\phi} \\ &= \sum_{a\in\Gamma} |\langle B^* M_{\chi_{D^r(a)}} H_f T e_a, e_a\rangle|^p \le C\|H_f\|^p_{S_p}. \end{aligned} \tag{4.36}$$

Now note that

$$\begin{aligned} \|f\|^p_{\mathrm{IDA}^{p,2,-2/p}_r} &= \int\limits_{\mathbb{C}} \rho^{-2} G_{2,r}(f)^p dA \\ &\le \sum_{a\in\Gamma} \int\limits_{D^r(a)} \rho(z)^{-2} G_{2,r}(f)(z)^p dA(z) \\ &\le \sum_{a\in\Gamma} \sup_{z\in D^r(a)} \rho(z)^{-2} G_{2,r}(f)(z)^p |D^r(a)| \\ &= C\sum_{a\in\Gamma} \rho(a)^{-2} G_{2,r}(f)(a)^p \rho(a)^2 \\ &= C\sum_{a\in\Gamma} G_{2,r}(f)(a)^p \\ &\le C\|H_f\|^p_{S_p}. \end{aligned} \tag{4.37}$$

Now since if Theorem 1.1 holds for some $r > 0$, it holds for any r, we are done with the proof for $p \ge 1$.

Now we finish the proof of Theorem 1.2 by showing that (1) $\implies$ (2) for $0 < p < 1$. Since $H_f \in S_p(F^2_\phi, L^2_\phi)$, it is in particular bounded. For $a \in \Gamma$ set

$$g_a = \begin{cases} \frac{\chi_{D^r(a)} fk_{2,a} - P_{a,r}(fk_{2,a})}{\|\chi_{D^r(a)} fk_{2,a} - P_{a,r}(fk_{2,a})\|} & \text{if } \|\chi_{D^r(a)} fk_{2,a} - P_{a,r}(fk_{2,a})\| \ne 0, \\ 0 & \text{if } \|\chi_{D^r(a)} fk_{2,a} - P_{a,r}(fk_{2,a})\| = 0. \end{cases} \tag{4.38}$$

Then similar as before, $\|g_a\| \le 1$, and $\langle g_a, g_b\rangle = 0$ for $a \ne b$. Let J be any finite subcollection of Γ, and $\{e_a\}_{a\in J}$ be an orthonormal set of L^2_ϕ. Define

$$A = \sum_{a\in J} e_a \otimes g_a : L^2_\phi \to L^2_\phi. \tag{4.39}$$

Then A is of finite rank and $\|A\| \le 1$. Similarly define

$$T = \sum_{a\in J} k_{2,a} \otimes e_a : L^2_\phi \to F^2_\phi. \tag{4.40}$$

Then as before, since Γ is an r-lattice and thus separated, there is a constant C such that $\|T\| \le C$. Then,

$$AH_fT = \sum_{a,\tau\in J} \langle H_f k_{2,\tau}, g_a\rangle e_a \otimes e_\tau = Y + Z, \tag{4.41}$$

where

$$Y = \sum_{a\in J} \langle H_f k_{2,a}, g_a\rangle e_a \otimes e_a, \qquad Z = \sum_{a,\tau\in J, a\neq\tau} \langle H_f k_{2,\tau}, g_a\rangle e_a \otimes e_\tau. \tag{4.42}$$

Note that

$$\begin{aligned}
\langle H_f k_{2,a}, g_a\rangle_{2,\phi} &= \langle f k_{2,a} - P(f k_{2,a}), g_a\rangle_{2,\phi} = \langle \chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a}), g_a\rangle_{2,\phi} \\
&= \|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\|_{2,\phi} \\
&= \left[\int_{\mathbb{C}} |\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})|^2 e^{-2\phi} dA \right]^{1/2} \\
&= \left[\int_{D^r(a)} |f k_{2,a} - P_{a,r}(f k_{2,a})|^2 e^{-2\phi} dA \right]^{1/2} \\
&= \left[\int_{D^r(a)} |f - \frac{P_{a,r}(f k_{2,a})}{k_{2,a}}|^2 |k_{2,a}|^2 e^{-2\phi} dA \right]^{1/2} \\
&\asymp \left[\frac{1}{|D^r(a)|} \int_{D^r(a)} |f - \frac{P_{a,r}(f k_{2,a})}{k_{2,a}}|^2 dA \right]^{1/2} \\
&\geq G_{2,r}(f)(a),
\end{aligned} \tag{4.43}$$

where in the line before the last line we have used (4.32) and $|D^r(a)| \asymp \rho(a)^2$. Thus,

$$\langle H_f k_{2,a}, g_a\rangle_{2,\phi} \geq C G_{2,r}(f)(a). \tag{4.44}$$

Therefore, there exists some N, independent of f and J such that

$$\|Y\|_{S_p}^p = \sum_{a\in J} \langle H_f k_{2,a}, g_a\rangle_{2,\phi}^p \geq N \sum_{a\in J} G_{2,r}(f)(a)^p. \tag{4.45}$$

On the other hand for $0 < p < 1$,

$$\|Z\|_{S_p}^p \leq \sum_{a,\tau\in J, a\neq\tau} \langle H_f k_{2,\tau}, g_a\rangle_{2,\phi}^p. \tag{4.46}$$

Let $Q_{a,r} : L^2(D^r(a), dA) \to A^2(D^r(a), dA)$ be the Bergman projection. Then $f k_{2,\tau} - P_{a,r}(f k_{2,\tau})$ and $P_{a,r}(f k_{2,\tau}) - k_{2,\tau} Q_{a,r} f$ are orthogonal, and by Parseval's identity,

$$\|f k_{2,\tau} - P_{a,r}(f k_{2,\tau})\|_{L^2(D^r(a), e^{-2\phi} dA)} \leq \|f k_{2,\tau} - k_{2,\tau} Q_{a,r}(f)\|_{L^2(D^r(a), e^{-2\phi} dA)}. \tag{4.47}$$

Note that by Lemma 2.3, there exist $C, \epsilon > 0$ such that

$$|K(w,z)| \le C\frac{e^{\phi(w)+\phi(z)}}{\rho(w)\rho(z)}e^{-(\frac{|z-w|}{\rho(z)})^{\epsilon}}. \tag{4.48}$$

Besides, by Lemma 6.8 in [18], we can see that given $R > 0$ and any finite sequence $\{a_j\}_{j=1}^n$ of different points in $\mathbb{C}$, it can be partitioned into subsequences such that any different points a_j and a_k in the same subsequence satisfy

$$|a_j - a_k| \ge R\min(\rho(a_j), \rho(a_k)). \tag{4.49}$$

So taking J to be a finite collection of Γ, we can choose an appropriately large $R > 0$ such that

$$|a-b| \ge R\min(\rho(a), \rho(b)), \quad \text{when } a, b \in J, a \ne b. \tag{4.50}$$

Putting everything together,

$$\begin{aligned}
|\langle H_f k_{2,\tau}, g_a\rangle| &= |\langle f k_{2,\tau} - P(f k_{2,\tau}), g_a\rangle| \\
&= |\langle f k_{2,\tau} - P(f k_{2,\tau}), \frac{\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})}{\|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\|}\rangle| \\
&= \frac{|\langle \chi_{D^r(a)} f k_{2,\tau} - P_{a,r}(f k_{2,\tau}), \chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\rangle|}{\|\chi_{D^r(a)} f k_{2,a} - P_{a,r}(f k_{2,a})\|} \\
&\le \|f k_{2,\tau} - P_{a,r}(f k_{2,\tau})\|_{L^2(D^r(a), e^{-2\phi}dA)} \\
&\overset{(4.47)}{\le} \|f k_{2,\tau} - k_{2,\tau} Q_{a,r}(f)\|_{L^2(D^r(a), e^{-2\phi}dA)} \\
&\le \sup_{\xi \in D^r(a)} |k_{2,\tau}(\xi) e^{-\phi}| \|f - Q_{a,r}(f)\|_{L^2(D^r(a), dA)} \\
&\overset{(4.48)}{\le} \sup_{\xi \in D^r(a)} \frac{C}{\rho(\xi)} e^{-(\frac{|\tau-\xi|}{\rho(\tau)})^{\epsilon}} \|f - Q_{a,r}(f)\|_{L^2(D^r(a), dA)} \\
&\simeq \frac{C}{\rho(a)} e^{-(\frac{|\tau-a|}{\rho(\tau)})^{\epsilon}} \|f - Q_{a,r}(f)\|_{L^2(D^r(a), dA)} \\
&\simeq \frac{C}{|D^r(a)|^{1/2}} \left[\int_{D^r(a)} |f - Q_{a,r}(f)|^2 dA\right]^{1/2} e^{-(\frac{|\tau-a|}{\rho(\tau)})^{\epsilon}} \\
&= C G_{2,r}(f)(a) e^{-(\frac{|\tau-a|}{\rho(\tau)})^{\epsilon}},
\end{aligned} \tag{4.51}$$

where in the last line we used the basic properties of Hilbert spaces. Therefore,

$$\begin{aligned}
\|Z\|_{S_p}^p &\overset{(4.46)}{\le} \sum_{a,\tau \in J, a \ne \tau} G_{2,r}(f)(a)^p e^{-(\frac{|\tau-a|}{\rho(\tau)})^{p\epsilon}} \\
&\overset{(4.49)}{\le} \sum_{a \in J} G_{2,r}(f)(a)^p \sum_{a,\tau \in J, a \ne \tau} e^{-(\frac{R\min(\rho(a),\rho(\tau))}{\rho(\tau)})^{p\epsilon}} \\
&\simeq \sum_{a \in J} G_{2,r}(f)(a)^p e^{-R^{p\epsilon}}.
\end{aligned} \tag{4.52}$$

Now we can pick some R large enough such that

$$\|Z\|_{S_p}^p \leq \frac{N}{4}\sum_{a\in J} G_{2,r}(f)(a)^p. \tag{4.53}$$

Using

$$\|Y\|_{S_p}^p \leq 2\|AH_fT\|_{S_p}^p + 2\|Z\|_{S_p}^p, \tag{4.54}$$

we have

$$N\sum_{a\in J} G_{2,r}(f)(a)^p \leq 2\|AH_fT\|_{S_p}^p + \frac{N}{2}\sum_{a\in J} G_{2,r}(f)(a)^p, \tag{4.55}$$

and since J is finite,

$$\begin{aligned} N\sum_{a\in J} G_{2,r}(f)(a)^p &\leq 2\|AH_fT\|_{S_p}^p \\ &\leq 4\|A\|_{L_\phi^2\to L_\phi^2}^p \|H_f\|_{S_p}^p \|T\|_{L_\phi^2\to L_\phi^2}^p \\ &\leq C\|H_f\|_{S_p}^p. \end{aligned} \tag{4.56}$$

Since C is independent of f and J,

$$\sum_{a\in\Gamma} G_{2,r}(f)(a)^p \leq C\|H_f\|_{S_p}^p. \tag{4.57}$$

The remaining of the proof is similar to (4.37) and we can conclude that for $0 < p < 1$,

$$\|f\|_{\mathrm{IDA}_r^{p,2,-2/p}} \leq C\|H_f\|_{S_p}^p. \quad \square \tag{4.58}$$

5. Simultaneous membership of H_f and $H_{\bar{f}}$ in S_p

In this section, we first define the space of functions of integral mean oscillation IMO and prove some of its basic properties. In particular, we prove that H_f and $H_{\bar{f}}$ are simultaneously in $S_p(F_\phi^2, L_\phi^2)$ with $0 < p < \infty$ if and only if the symbol f satisfies a suitable IMO condition (see Theorem 1.3).

Lemma 5.1. *Let $0 < p < \infty$ and $r > 0$. Then for $f \in L_{loc}^2$, $f \in \mathrm{IMO}_r^{p,2,\alpha}$ if and only if there exists a continuous function $c(z)$ on $\mathbb{C}$ such that*

$$\rho^\alpha \left(\frac{1}{|D^r(z)|} \int\limits_{D^r(z)} |f(w) - c(z)|^2 dA(w) \right)^{1/2} \in L^p \tag{5.1}$$

Proof. This proof is similar to the proof of Proposition 2.4 in [13]. We can similarly extend the proposition to the case $0 < p < 1$, and the doubling weights by introducing ρ as the following. First note that if $f \in \mathrm{IMO}_r^{p,2,\alpha}$, then (5.1) holds with $c(z) = \hat{f}_r(z)$ which is continuous for $z \in \mathbb{C}$. Conversely, assume that (5.1) holds. By Minkowski inequality,

$$\rho^\alpha(z) MO_{2,r}(f)(z) \leq \rho^\alpha \big(\tfrac{1}{|D^r(z)|} \int_{D^r(z)} |f - c(z)|^2 dA \big)^{1/2} + \rho^\alpha |\hat{f}_r(z) - c(z)|. \tag{5.2}$$

By Hölder's inequality,

$$\rho^\alpha|\hat{f}_r(z)-c(z)| \leq \rho^\alpha\big(\tfrac{1}{|D^r(z)|}\int_{D^r(z)}|f-c(z)|^2dA\big)^{1/2} \in L^p \quad \text{by (5.1)}. \tag{5.3}$$

Hence, using (5.2) and (5.3) we can see that $f \in \mathrm{IMO}_r^{p,2,\alpha}$. □

Proposition 5.2. *Let $0 < p \leq \infty$, $r > 0$, and $f \in L^2_{loc}$. If for each $z \in \mathbb{C}$, there exist $h_1, h_2 \in H(D^r(z))$ such that*

$$\rho^\alpha(z)\big(\frac{1}{|D^r(z)|}\int_{D^r(z)}|f-h_1|^2dA\big)^{1/2} \in L^p,$$

and

$$\rho^\alpha(z)\big(\frac{1}{|D^r(z)|}\int_{D^r(z)}|\bar{f}-h_2|^2dA\big)^{1/2} \in L^p, \tag{5.4}$$

then $f \in \mathrm{IMO}_r^{p,2,\alpha}$.

Proof. The proof is a more detailed version of the proof of Proposition 2.5 in [13], extended to the case of doubling Fock spaces. For $f \in L^2_{loc}$, recall that

$$\big(\widehat{|f|^2}_r(z)\big)^{1/2} = \big(\frac{1}{|D^r(z)|}\int_{D^r(z)}|f|^2dA\big)^{1/2}. \tag{5.5}$$

By the triangle inequality and using (5.4),

$$\rho^\alpha\big(\widehat{|\tfrac{f+\bar{f}}{2}-\tfrac{h_1+h_2}{2}|^2}_r(z)\big)^{1/2} \leq \rho^\alpha\big(\widehat{|\tfrac{f-h_1}{2}|^2}_r(z)\big)^{1/2} + \rho^\alpha\big(\widehat{|\tfrac{\bar{f}-h_2}{2}|^2}_r(z)\big)^{1/2} \in L^p. \tag{5.6}$$

Since $f+\bar{f}$ and ρ^α are real-valued, we can conclude that

$$\rho^\alpha\big(\widehat{|\operatorname{Im}\tfrac{h_1+h_2}{2}|^2}_r(z)\big)^{1/2} \in L^p. \tag{5.7}$$

As in the proof of the Proposition 2.5 in [13], we know that if $v : D^r(z) \to \mathbb{R}$ is harmonic, there exists a harmonic function u such that $u+iv \in H(D^r(z))$ and

$$\|u-u(z)\|_{L^q(D^r(z),dA)} \leq C\|v\|_{L^q(D^r(z),dA)}, \tag{5.8}$$

for all $0 < q < \infty$.

Taking $q = 2$ in (5.8), and since $h_1 + h_2 \in H(D^r(z))$,

$$\big(\widehat{|\operatorname{Re}\tfrac{h_1+h_2}{2}-\operatorname{Re}\tfrac{h_1+h_2}{2}(z)|^2}_r(z)\big)^{1/2} \leq C\big(\widehat{|\operatorname{Im}\tfrac{h_1+h_2}{2}|^2}_r(z)\big)^{1/2}. \tag{5.9}$$

Thus,

$$\begin{aligned}
\rho^\alpha\big(\widehat{|\tfrac{f+\bar{f}}{2}-\operatorname{Re}\tfrac{h_1+h_2}{2}(z)|^2}_r(z)\big)^{1/2} &\leq \rho^\alpha\big(\widehat{|\tfrac{f+\bar{f}}{2}-\operatorname{Re}\tfrac{h_1+h_2}{2}|^2}_r(z)\big)^{1/2} \\
&\quad + \rho^\alpha\big(\widehat{|\operatorname{Re}\tfrac{h_1+h_2}{2}-\operatorname{Re}\tfrac{h_1+h_2}{2}(z)|^2}_r(z)\big)^{1/2} \\
&\leq \rho^\alpha\big(\widehat{|\tfrac{f+\bar{f}}{2}-\tfrac{h_1+h_2}{2}|^2}_r(z)\big)^{1/2} \\
&\quad + C\rho^\alpha\big(\widehat{|\operatorname{Im}\tfrac{h_1+h_2}{2}|^2}_r(z)\big)^{1/2} \in L^p,
\end{aligned} \tag{5.10}$$

where the first term in the last line is in L^p by (5.6), and the second term is in L^p by (5.7). Hence,

$$\rho^\alpha\big(\widehat{|\tfrac{f+\bar f}{2} - \operatorname{Re}\tfrac{h_1+h_2}{2}(z)|^2}_r(z)\big)^{1/2} \in L^p. \tag{5.11}$$

Similar to (5.6), (5.7), and (5.8), and applying (5.4), we have

$$\rho^\alpha\big(\widehat{|\tfrac{f-\bar f}{2} - \tfrac{h_1-h_2}{2}|^2}_r(z)\big)^{1/2} \le \rho^\alpha\big(\widehat{|\tfrac{f-h_1}{2}|^2}_r(z)\big)^{1/2} + \rho^\alpha\big(\widehat{|\tfrac{\bar f-h_2}{2}|^2}_r(z)\big)^{1/2} \in L^p \tag{5.12}$$

Since $\frac{f-\bar f}{2}$ is completely imaginary, we can conclude that

$$\rho^\alpha\big(\widehat{|\operatorname{Re}\tfrac{h_1-h_2}{2}|^2}_r(z)\big)^{1/2} \in L^p. \tag{5.13}$$

We can exchange u and v in (5.8), and therefore,

$$\big(\widehat{|\operatorname{Im}\tfrac{h_1-h_2}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\big)^{1/2} \le C\big(\widehat{|\operatorname{Re}\tfrac{h_1-h_2}{2}|^2}_r(z)\big)^{1/2}. \tag{5.14}$$

Thus by (5.12) and (5.13),

$$\begin{aligned}\rho^\alpha\big(\widehat{|\tfrac{f-\bar f}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\big)^{1/2} &\le \rho^\alpha\big(\widehat{|\tfrac{f-\bar f}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}|^2}_r(z)\big)^{1/2} \\ &\quad + \rho^\alpha\big(\widehat{|\operatorname{Im}\tfrac{h_1-h_2}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\big)^{1/2} \\ &\le \rho^\alpha\big(\widehat{|\tfrac{f-\bar f}{2} - \tfrac{h_1-h_2}{2}|^2}_r(z)\big)^{1/2} \\ &\quad + C\rho^\alpha\big(\widehat{|\operatorname{Re}\tfrac{h_1-h_2}{2}|^2}_r(z)\big)^{1/2} \in L^p. \end{aligned} \tag{5.15}$$

Hence, analogous to (5.11),

$$\rho^\alpha\big(\widehat{|\tfrac{f-\bar f}{2} - \operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\big)^{1/2} \in L^p. \tag{5.16}$$

Choose $c(z) = \operatorname{Re}\frac{h_1+h_2}{2}(z) + i\operatorname{Im}\frac{h_1-h_2}{2}(z)$. Then by (5.11) and (5.16),

$$\rho^\alpha\big(\widehat{|f-c(z)|^2}_r(z)\big)^{1/2} \in L^p, \tag{5.17}$$

which is equivalent to

$$\rho^\alpha\Big(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f-c(z)|^2 dA\Big)^{1/2} \in L^p. \tag{5.18}$$

Thus by Lemma 5.1 we can conclude that $f \in \mathrm{IMO}_r^{p,2,\alpha}$. $\square$

Lemma 5.3. *Let $0 < p \le \infty$. Then for $f \in L^2_{loc}$, $f \in \mathrm{IDA}_r^{p,2,\alpha}$ and $\bar f \in \mathrm{IDA}_r^{p,2,\alpha}$ if and only if $f \in \mathrm{IMO}_r^{p,2,\alpha}$.*

Proof. First, we show that

$$\|f\|_{\mathrm{IMO}_r^{p,2,\alpha}} = \|\rho^\alpha MO_{2,r}(f)\|_{L^p} \lesssim \|f\|_{\mathrm{IDA}_r^{p,2,\alpha}} + \|\bar f\|_{\mathrm{IDA}_r^{p,2,\alpha}}. \tag{5.19}$$

Note that by Lemma 3.1, there exists $h_1, h_2 \in H(D^r(z))$ such that

$$G_{2,r}(f)(z) = \big(\widehat{|f-h_1|^2}_r(z)\big)^{1/2}, \quad \text{and} \quad G_{2,r}(\bar f)(z) = \big(\widehat{|\bar f-h_2|^2}_r(z)\big)^{1/2}. \tag{5.20}$$

Taking $c(z)$ as in the proof of the previous lemma, and using (5.10), (5.15), (5.6), and (5.12),

$$\begin{aligned}\rho^\alpha\big(\widehat{|f-c(z)|^2}_r(z)\big)^{1/2} &= \rho^\alpha\big(\widehat{|\tfrac{f+\bar f}{2} - \operatorname{Re}\tfrac{h_1+h_2}{2}(z) + \tfrac{f-\bar f}{2} - i\operatorname{Im}\tfrac{h_1-h_2}{2}(z)|^2}_r(z)\big)^{\frac12}\\ &\le C\rho^\alpha\big(G_{2,r}(f)(z) + G_{2,r}(\bar f)(z)\big)\\ &+ C\rho^\alpha\big\{\big(\widehat{|\operatorname{Im}\tfrac{h_1+h_2}{2}|^2}_r(z)\big)^{1/2} + \big(\widehat{|\operatorname{Re}\tfrac{h_1-h_2}{2}|^2}_r(z)\big)^{1/2}\big\}.\end{aligned} \tag{5.21}$$

Note that since L^2 is a Hilbert space, we can set $h_1 = Q_{z,r}(f)$ and $h_2 = Q_{z,r}(\bar f)$. Then the linearity of the Bergman projection $Q_{z,r} : L^2(D^r(z), dA) \to A^2(D^r(z), dA)$ implies that the last two terms are zero. Thus,

$$\rho^\alpha\big(\widehat{|f-c(z)|^2}_r(z)\big)^{1/2} \le C\rho^\alpha\big(G_{2,r}(f)(z) + G_{2,r}(\bar f)(z)\big). \tag{5.22}$$

Hence,

$$\rho^\alpha MO_{2,r}(f)(z) \le \rho^\alpha\Big(\frac{1}{|D^r(z)|}\int\limits_{D^r(z)} |f-c(z)|^2 dA\Big)^{1/2} + \rho^\alpha|\hat f_r(z) - c(z)|. \tag{5.23}$$

By Hölder's inequality,

$$|\hat f_r(z) - c(z)| \le \big(\tfrac{1}{|D^r(z)|}\textstyle\int_{D^r(z)} |f-c(z)|^2 dA\big)^{1/2}. \tag{5.24}$$

Applying this to (5.23), and using (5.22), we get

$$\rho^\alpha MO_{2,r}(f)(z) \le C\rho^\alpha\big\{G_{2,r}(f)(z) + G_{2,r}(\bar f)(z)\big\}. \tag{5.25}$$

Taking the L^p-norms of both sides we can conclude that for $0 < p \le \infty$,

$$\|f\|_{\mathrm{IMO}_r^{p,2,\alpha}} \lesssim \|f\|_{\mathrm{IDA}_r^{p,2,\alpha}} + \|\bar f\|_{\mathrm{IDA}_r^{p,2,\alpha}}. \tag{5.26}$$

For the inverse inequality, note that using the definition, it is immediate to see that $f \in \mathrm{IMO}_r^{p,2,\alpha}$ if and only if $\bar f \in \mathrm{IMO}_r^{p,2,\alpha}$. Moreover, $\hat f_r(z)$ is a constant, and therefore holomorphic. So by definition, $\|f\|_{\mathrm{IDA}_r^{p,2,\alpha}} \le \|f\|_{\mathrm{IMO}_r^{p,2,\alpha}}$. Similarly, $\|\bar f\|_{\mathrm{IDA}_r^{p,2,\alpha}} \le \|\bar f\|_{\mathrm{IMO}_r^{p,2,\alpha}} = \|f\|_{\mathrm{IMO}_r^{p,2,\alpha}}$, and we are done. □

We can now give the proof of Theorem 1.3, which shows that both H_f and $H_{\bar f}$ are in S_p if and only if $f \in \mathrm{IMO}_r^{p,2,-2/p}$, where $1 < p < \infty$.

Proof of Theorem 1.3. By Theorem 1.2, $H_f \in S_p$ if and only if $f \in \mathrm{IDA}_r^{p,2,-2/p}$ for some (equivalent any) $r > 0$. Similarly, $H_{\bar f} \in S_p$ if and only if $\bar f \in \mathrm{IDA}_r^{p,2,-2/p}$. An application of Lemma 5.3 shows that this is equivalent to $f \in \mathrm{IMO}_r^{p,2,-2/p}$, for some (equivalent any) $r > 0$. Further, the norm estimates in (1.15) follow from (1.12) and (5.19). □

As mentioned in the introduction, we obtain the following result as a consequence of Theorem 1.3.

Theorem 5.4. *Let f be a non-constant entire function and F^2_ϕ be a doubling Fock space. Then $H_{\bar f}$ is not in $S_2(F^2_\phi, L^2_\phi)$.*

Proof. Since f is holomorphic, $H_f = 0$, and thus belongs to the Hilbert-Schmidt class. Applying Theorem 1.3, it is enough to show that $f \notin \mathrm{IMO}_1^{2,2,-1}$. First note that $\bar{f}$ is harmonic on $D^1(z)$ and by the mean-value property of harmonic functions,

$$\widehat{f}_1(z) = \frac{1}{|D^1(z)|} \int_{D^1(z)} f dA = f(z).$$

By the Cauchy estimate,

$$\begin{aligned} MO_{2,1}(f)(z) &= \left(\tfrac{1}{|D^1(z)|} \int_{D^1(z)} |f(w) - f(z)|^2 dA(w) \right)^{1/2} \\ &\geq C|\partial f(z)|\rho(z). \end{aligned}$$

Hence,

$$\begin{aligned} \|f\|_{\mathrm{IMO}_1^{2,2,-1}} &= \int_{\mathbb{C}} \rho(z)^{-2} MO_{2,1}(f)(z)^2 dA(z) \\ &\geq C \int_{\mathbb{C}} \rho(z)^{-2} |\partial f(z)|^2 \rho(z)^2 dA(z). \end{aligned}$$

So, since f is entire and non-constant, it follows that $f \notin \mathrm{IMO}_1^{2,2,-1}$, and thus $H_{\bar{f}}$ is not Hilbert-Schmidt. □

6. Berger-Coburn phenomenon for doubling Fock spaces

This section contains the proofs of Theorems 1.4 and 1.6. We start with the proof of the Berger-Coburn phenomenon for Hilbert-Schmidt Hankel operators, that is, we show that for $f \in L^\infty$, H_f is Hilbert-Schmidt if and only if $H_{\bar{f}}$ is Hilbert-Schmidt.

Proof of Theorem 1.4. Let $H_f \in S_2$. By the assumption, $f \in L^\infty$, and in particular $f \in L^2_{loc}$. Then by Theorem 1.2, $f \in \mathrm{IDA}_r^{2,2,-1}$ for some (equivalent any) $r > 0$, and

$$\|f\|_{\mathrm{IDA}_r^{2,2,-1}} \simeq \|H_f\|_{S_2} < \infty. \tag{6.1}$$

Decompose $f = f_1 + f_2$ as in (1.10). Thus $f_1 \in \mathcal{C}^2(\mathbb{C})$ and

$$|\bar{\partial} f_1| + (\widehat{|\bar{\partial} f_1|^2}_r)^{1/2} + \rho^{-1}(\widehat{|f_2|^2}_r)^{1/2} \in L^2. \tag{6.2}$$

Then the definition

$$\rho^{-1}(z)(\widehat{|f_2|^2}_r(z))^{1/2} = \rho^{-1}(z)\Big(\frac{1}{|D^r(z)|} \int_{D^r(z)} |f_2|^2 dA\Big)^{1/2} \tag{6.3}$$

implies that

$$\rho^{-1}(\widehat{|f_2|^2}_r)^{1/2} = \rho^{-1}(\widehat{|\bar{f}_2|^2}_r)^{1/2} \in L^2. \tag{6.4}$$

By (1.11) and (1.12), $H_{\bar{f}_2} \in S_2$. Indeed,

$$\|H_{\bar{f}_2}\|_{S_2} \overset{(1.12)}{=} \|\bar{f}_2\|_{\mathrm{IDA}^{2,2,-1}_r} \overset{(1.11)}{\lesssim} \|\rho^{-1}(\widehat{|\bar{f}_2|^2}_r)^{1/2}\|_{L^2}$$
$$\overset{(6.4)}{=} \|\rho^{-1}(\widehat{|f_2|^2}_r)^{1/2}\|_{L^2} \overset{(1.11)}{\lesssim} \|f\|_{\mathrm{IDA}^{2,2,-1}_r}. \tag{6.5}$$

To show that $\|H_{\bar{f}_1}\|_{S_2} \lesssim \|f\|_{\mathrm{IDA}^{2,2,-1}_r}$, we need to follow a more complicated argument, inspired by the proof of Theorem 1.2 in [10]. Let $\{a_j\}_{j=1}^\infty$ be a fixed $m_1 r$-lattice for some $m_1 \in (0,1)$ and $r > 0$. Choose a partition of unity $\{\psi_j\}_{j=1}^\infty$ subordinate to $\{D^{m_1 r}(a_j)\}$ as in (3.9). By Lemma 3.1 there exists $h_j \in H(D^r(a_j))$ such that

$$\left(\widehat{|f-h_j|^2}_r(a_j)\right)^{1/2} = G_{2,r}(f)(a_j), \quad \text{and} \sup_{z\in D^{m_1 r}(a_j)} |h_j(z)| \lesssim \|f\|_{L^\infty}. \tag{6.6}$$

Now we get back to the decomposition $f = f_1 + f_2$ as in (1.10) with $f_1 = \sum_{j=1}^\infty h_j \psi_j$. Without loss of generality we can assume $\psi_j = \bar{\psi}_j$ for all $j \geq 1$. Since we assumed that f is bounded, $f_1 \in L^\infty$ and moreover

$$\bar{\partial}\bar{f}_1 = \sum_{j=1}^\infty \bar{h}_j \bar{\partial}\psi_j + \sum_{j=1}^\infty \psi_j \bar{\partial}\bar{h}_j = F + H, \tag{6.7}$$

for $F = \sum_{j=1}^\infty \bar{h}_j \bar{\partial}\psi_j$ and $H = \sum_{j=1}^\infty \psi_j \bar{\partial}\bar{h}_j$. Similar to (3.13) one has

$$|F(z)| = \rho^{-1}(z)\rho(z)|\sum_{j=1}^\infty \bar{h}_j \bar{\partial}\psi_j| = \rho^{-1}(z)\rho(z)|\sum_{j=1}^\infty \bar{h}_j \bar{\partial}\psi_j - \sum_{j=1}^\infty \bar{h}_1 \bar{\partial}\psi_j|$$
$$\leq \rho^{-1}(z)\rho(z)\sum_{j=1}^\infty |\bar{h}_j(z) - \bar{h}_1(z)||\bar{\partial}\psi_j(z)| \leq C\rho^{-1}(z)G_{2,r}(f)(z). \tag{6.8}$$

Besides,

$$\|H\|_{L^2} \leq \|\bar{\partial}\bar{f}_1\|_{L^2} + \|F\|_{L^2}. \tag{6.9}$$

By (6.8),

$$\|F\|_{L^2} \leq \|f\|_{\mathrm{IDA}^{2,2,-1}_r}. \tag{6.10}$$

Lemma 7.1 in [9] implies that

$$\|\bar{\partial}\bar{f}_1\|_{L^2} = \|\partial f_1\|_{L^2} \leq C\|\bar{\partial} f_1\|_{L^2} \leq C\|f\|_{\mathrm{IDA}^{2,2,-1}_r}, \tag{6.11}$$

where the last inequality is obtained by multiplying both sides of (3.12) with ρ^{-1}. Hence, we can conclude that

$$\|H\|_{L^2} \lesssim \|f\|_{\mathrm{IDA}^{2,2,-1}_r}. \tag{6.12}$$

Note that for $m_1, m_2 \in (0,1)$,

$$\|H_{\bar{f}_1}\|^2_{S_2} \simeq \|\bar{f}_1\|^2_{\mathrm{IDA}^{2,2,-1}_r} \overset{(1.10)}{\leq} C \int_{\mathbb{C}} \left[(\widehat{|\bar{\partial}\bar{f}_1|^2}_{m m_2 r})^{1/2}\right]^2 dA$$
$$\lesssim \int_{\mathbb{C}} \left[(\widehat{|F|^2}_{m_1 m_2 r})^{1/2}\right]^2 dA + \int_{\mathbb{C}} \left[(\widehat{|H|^2}_{m_1 m_2 r})^{1/2}\right]^2 dA, \tag{6.13}$$

where for the last inequality we used the equivalence $\rho(w) \simeq \rho(z)$ for $w \in D^{m_1 m_2 r}(z)$ and (6.7). Note that using (6.8) one has

$$\int_{\mathbb{C}} \left[(\widehat{|F|^2}_{m_1 m_2 r})^{1/2}\right]^2 dA \lesssim \|f\|^2_{\mathrm{IDA}^{2,2,-1}_r}, \tag{6.14}$$

and thus we are left to compute $\int_{\mathbb{C}} \left[(\widehat{|H|^2}_{m_1 m_2 r})^{1/2}\right]^2 dA$. Let $z \in D^r(a_j) \cap D^r(a_k)$. Since $|\bar\partial(\bar h_k - \bar h_j)| = |\partial(h_k - h_j)|$, applying the Cauchy estimate for the boundary of the disk $D^{m_1 m_2 r}(z)$ of radius $m_1 m_2 r \rho(z)$ and Hölder's inequality, we obtain the following.

$$|\bar\partial(\bar h_k(z) - \bar h_j(z))| \le \frac{C}{\rho(z)} \Big\{ \int_{D^{m_1 m_2 r}(z)} |\bar h_k(w) - \bar h_j(w)|^2 dA \Big\}^{1/2}. \tag{6.15}$$

Using $|\bar h_k - \bar h_j|^2 = |(f - \bar h_k) - (f - \bar h_j)|^2 \le |f - \bar h_k|^2 + |f - \bar h_j|^2$, and the fact that h_k and h_j are holomorphic, we get

$$\begin{aligned} |\bar\partial(\bar h_k(z) - \bar h_j(z))| &\le \frac{C}{\rho(z)} \big(G_{2,m_1 m_2 r}(f)(a_k) + G_{2,m_1 m_2 r}(f)(a_j) \big) \\ &\le \frac{C}{\rho(z)} G_{2,R}(f)(z), \end{aligned} \tag{6.16}$$

for some $R > m_1 m_2 r$. Recalling H as in (6.7),

$$H + \sum_{j=1}^{\infty} \psi_j \bar\partial(\bar h_k - \bar h_j) = H + \sum_{j=1}^{\infty} \psi_j \bar\partial \bar h_k - H. \tag{6.17}$$

Since $\{\psi_j\}_{j=1}^{\infty}$ is a partition of unity and therefore $\sum_{j=1}^{\infty} \psi_j = 1$,

$$\bar\partial \bar h_k = \sum_{j=1}^{\infty} \psi_j \bar\partial(\bar h_k - \bar h_j) + H. \tag{6.18}$$

Hence,

$$\begin{aligned} |\bar\partial \bar h_k(z)|^2 &\lesssim \Big| \sum_{j=1}^{\infty} \psi_j(z) \bar\partial(\bar h_k(z) - \bar h_j(z)) \Big|^2 + \big|H(z)\big|^2 \\ &\lesssim \sum_{j \in D^{m_1 r}(a_j)} \psi_j(z) |\bar\partial(\bar h_k(z) - \bar h_j(z))|^2 + |H(z)|^2 \\ &\lesssim \big(\rho^{-1}(z) G_{2,R}(f)(z)\big)^2 + |H(z)|^2, \end{aligned} \tag{6.19}$$

where the last inequality follows from (6.16). For $z \in D^{m_1 r}(a_k)$, notice that $D^{m_1 m_2 r}(z) \subset D^{m_1 r}(a_k)$ for some $m_2 \in (0,1)$. Then by subharmonicity,

$$\begin{aligned} |\bar\partial \bar h_k(z)|^2 &\le \frac{1}{|D^{m_1 m_2 r}(z)|} \int_{D^{m_1 m_2 r}(z)} |\bar\partial \bar h_k(w)|^2 dA(w) \\ &\overset{(6.19)}{\lesssim} \frac{1}{|D^{m_1 m_2 r}(z)|} \int_{D^{m_1 m_2 r}(z)} \left[\big|\rho^{-1}(w) G_{2,R}(f)(w)\big|^2 + \big|H(w)\big|^2 \right] dA(w) \end{aligned}$$

$$\lesssim (\rho^{-1}(z))^2 G_{2,\tilde{R}}(f)(z)^2 + \widehat{|H|^2}_{m_1 m_2 r}(z), \tag{6.20}$$

for some $\tilde{R} > R$.

Now for $z \in \mathbb{C}$, there exists $w' \in \overline{D^{m_1 m_2 r}(z)}$ such that

$$\begin{aligned} \big[(\widehat{|H|^2}_{m_1 m_2 r}(z))^{1/2}\big]^2 &\leq \max\{|H(w)|^2 : w \in \overline{D^{m_1 m_2 r}(z)}\} \\ &= \big|\sum_{k=1}^{\infty} \psi_k(w')\bar{\partial}\bar{h}_k(w')\big|^2, \end{aligned} \tag{6.21}$$

where the first inequality comes from integration on a bounded domain. Note that $G_{2,\tilde{R}}(f)(w')^2 \lesssim G_{2,s}(f)(z)^2$ for some $s > \tilde{R}$, and

$$\big[(\widehat{|H|^2}_{m_1 m_2 r}(w'))^{1/2}\big]^2 \leq \big[(\widehat{|H|^2}_{m_1 r}(z))^{1/2}\big]^2, \tag{6.22}$$

and we can conclude that

$$\begin{aligned} \big[(\widehat{|H|^2}_{m_1 m_2 r}(z))^{1/2}\big]^2 &\overset{(6.21)}{\leq} \big|\sum_{k=1}^{\infty} \psi_k(w')\bar{\partial}\bar{h}_k(w')\big|^2 \\ &\overset{(6.20)}{\lesssim} \sum_{k,\psi_k(w')\neq 0} \psi_k(w')\Big\{(\rho^{-1}(w'))^2 G_{2,\tilde{R}}(f)(w')^2 \\ &\qquad + \widehat{|H|^2}_{m_1 m_2 r}(w')\Big\} \\ &\overset{(6.22)}{\lesssim} \big(\rho^{-1}(z))G_{2,s}(f)(z)\big)^2 + \widehat{|H|^2}_{m_1 r}(z). \end{aligned} \tag{6.23}$$

Hence as mentioned in (6.13), and applying Theorem 1.1,

$$\begin{aligned} \|H_{\bar{f}_1}\|_{S_2}^2 &\lesssim \|f\|^2_{\mathrm{IDA}^{2,2,-1}_s} + \int_{\mathbb{C}} \big[(\widehat{|H|^2}_{m_1 m_2 r}(z))^{1/2}\big]^2 dA(z) \\ &\lesssim \|f\|^2_{\mathrm{IDA}^{2,2,-1}_s} + \int_{\mathbb{C}} \big(\rho^{-1}(z))G_{2,s}(f)(z)\big)^2 dA(z) + \int_{\mathbb{C}} \widehat{|H|^2}_{m_1 r}(z) dA(z) \\ &\lesssim \|f\|^2_{\mathrm{IDA}^{2,2,-1}_s} + \int_{\mathbb{C}} |H|^2 dA \\ &\lesssim \|f\|^2_{\mathrm{IDA}^{2,2,-1}_s}, \end{aligned} \tag{6.24}$$

where in the last line we have used (6.12).
This together with (6.5) implies that

$$\|H_{\bar{f}}\|_{S_2} \lesssim \|H_f\|_{S_2}. \tag{6.25}$$

We are done since the proof is symmetric for f and $\bar{f}$. $\square$

We make the following remark related to the Berger-Coburn phenomenon for other values of p.

Remark 6.1. For $1 < p < \infty$ we say that ω is a Muckenhoupt weight and write $\omega \in A_p$ if there is a constant $C > 0$ such that for any disk $B \subset \mathbb{C}$, we have

$$\left(\frac{1}{|B|}\int_B \omega dA\right)\left(\frac{1}{|B|}\int_B \omega^{-q/p} dA\right)^{p/q} \leq C < \infty, \tag{6.26}$$

where q is the Hölder conjugate of p and $|B|$ is the Lebesgue measure of B. As shown in [4], if $\omega \in A_p$ and $1 < p < \infty$, then the Ahlfors-Beurling operator

$$\mathcal{I}(f)(z) = p.v. - \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2} dA(z) \tag{6.27}$$

is bounded on $L^p(\omega)$. Hence, similarly to the proof of Lemma 7.1 in [9], we can show that when f is bounded,

$$\|\partial f\|_{L^p(\omega)} \leq C\|\bar{\partial} f\|_{L^p(\omega)}, \tag{6.28}$$

where C is a constant depending only on p.

To generalize Theorem 1.4 to the other values of $1 < p < \infty$, our approach would require only one additional ingredient that $\omega = \rho^{p-2}$ is a Muckenhoupt weight (see (6.11)). However, we have not been able to prove this condition and also note that Lemma 2.1 does not seem to help because the constants c_r in (2.2) are not bounded in general.

Next, we consider the case $0 < p \leq 1$. Recently Xia [21] defined the following simple function

$$f(z) := \begin{cases} \frac{1}{z} & \text{if } |z| \geq 1, \\ 0 & \text{if } |z| < 1, \end{cases} \tag{6.29}$$

and used it to show that the Berger-Coburn phenomenon does not hold for trace class Hankel operators on the classical Fock space. Hu and Virtanen [12] noticed that when $0 < p \leq 1$ the same example shows that there is no Berger-Coburn for Schatten class Hankel operators on generalized Fock spaces. Here we use Xia's example again to prove that the Berger-Coburn phenomenon fails for some $S_p(F^2_\phi, L^2_\phi)$ while it remains open whether it fails for the remaining doubling Fock spaces.

Proof of Theorem 1.6. To prove the theorem, we use Theorems 1.2 and 1.3. The idea is to find a bounded function f with $f \in \mathrm{IDA}^{p,2,-2/p}_r$ such that $f \notin \mathrm{IMO}^{p,2,-2/p}_r$ for some (equivalent any) $r > 0$. Note that by remark 1 in [16], there are constants $C, \eta > 0$, and $0 \leq \beta < 1$ such that for $|z| > 1$,

$$C^{-1}|z|^{-\eta} \leq \rho(z) \leq C|z|^\beta. \tag{6.30}$$

Let f be as in (6.29). By Theorem 1.1, the definition of $\mathrm{IDA}^{p,2,-2/p}_r$ is independent of r. So for simplicity, we set $r = 1$. It is easy to see that for a large enough $R > 0$, and $|z| \geq R$, f is holomorphic in $D^1(z) = D(z, \rho(z))$, and hence trivially $G_{2,1}(f_\beta)(z) = 0$. Indeed, one can see that for $|z| \geq R$, $D^1(z) \cap D(0,1) = \emptyset$. Moreover, for all $|z| < R$, there is a constant C such that

$$G_{2,1}(f)(z) < C, \tag{6.31}$$

as f is bounded in the bounded domain $D^1(z)$. Thus,

$$\|f\|^p_{\mathrm{IDA}_1^{p,2,-2/p}} = \|\rho^{-2/p} G_{2,1}(f)\|^p_{L^p} = \int_{\mathbb{C}} \rho^{-2} G_{2,1}(f)^p dA$$
$$\leq C \int_{|z|<R} \rho^{-2} dA < \infty. \tag{6.32}$$

Indeed, by Theorem 14 in [16], there is a smooth function ψ, where $\Delta\psi dA$ is doubling and $\Delta\psi \simeq \rho_\psi^{-2} \simeq \rho^{-2}$. Hence,

$$\int_{|z|<R} \rho^{-2} dA \simeq \int_{|z|<R} \Delta\psi dA < \infty, \tag{6.33}$$

as the doubling measures are locally finite. So by (6.32), $f \in \mathrm{IDA}_1^{p,2,-2/p}$, and Theorem 1.2 implies that $H_f \in S_p$.

To show that $H_{\bar{f}} \notin S_p$, note that if $|z| \geq R$, $\bar{f}$ is harmonic on $D^1(z)$ and by the mean-value property of harmonic functions,

$$\widehat{\bar{f}_1}(z) = \frac{1}{|D^1(z)|} \int_{D^1(z)} \bar{f} dA = \bar{f}(z). \tag{6.34}$$

Moreover, by definition, $MO_{2,r}(f)(z) = MO_{2,r}(\bar{f})(z)$, and thus for $|z| \geq R$,

$$MO_{2,1}(f)(z) = \left(\frac{1}{|D^1(z)|} \int_{D^1(z)} |\bar{f}(w) - \bar{f}(z)|^2 dA(w) \right)^{1/2}$$
$$= \left(\frac{1}{|D^1(z)|} \int_{D^1(z)} |\frac{1}{\bar{w}} - \frac{1}{\bar{z}}|^2 dA(w) \right)^{1/2}$$
$$= \left(\frac{1}{|D^1(z)|} \int_{D^1(z)} \frac{|w-z|^2}{|zw|^2} dA(w) \right)^{1/2}. \tag{6.35}$$

For $w \in D^1(z)$, we can write $w = z + re^{i\theta}$ where $0 \leq r < \rho(z)$ and $0 \leq \theta < 2\pi$. Therefore,

$$\int_{D^1(z)} \frac{|w-z|^2}{|zw|^2} dA(w) = \tfrac{1}{|z|^2} \int_0^{\rho(z)} r^3 \int_0^{2\pi} \tfrac{d\theta dr}{|z+re^{i\theta}|^2} \tag{6.36}$$

Let $z = |z|e^{i\psi}$. Then

$$\int_0^{2\pi} \frac{d\theta}{|z+re^{i\theta}|^2} = \int_0^{2\pi} \tfrac{d\theta}{||z|+re^{i\theta}|^2} = \int_0^{2\pi} \tfrac{d\theta}{|z|^2+r^2+2|z|r\cos\theta}. \tag{6.37}$$

Defining $y = \frac{r}{|z|}$,

$$\frac{1}{|z|^2}\int_0^{\rho(z)}\int_0^{2\pi}\frac{r^3 d\theta dr}{|z|^2+r^2+2|z|r\cos\theta}=\frac{1}{|z|^2}\int_0^{\frac{\rho(z)}{|z|}}\int_0^{2\pi}\frac{y^3|z|^4 d\theta dy}{|z|^2+y^2|z|^2+2|z|^2 y\cos\theta}$$
$$=\int_0^{\frac{\rho(z)}{|z|}}\frac{y^3}{2y}\int_0^{2\pi}\frac{d\theta dy}{\frac{1+y^2}{2y}+\cos\theta}.$$

Let $x=\frac{1+y^2}{2y}$. Then

$$\int_0^{2\pi}\frac{d\theta}{\frac{1+y^2}{2y}+\cos\theta}=\int_0^{2\pi}\frac{d\theta}{x+\cos\theta}. \tag{6.38}$$

Taking $t=\tan\frac{\theta}{2}$, we have $\theta=2\tan^{-1}(t)$, $d\theta=\frac{2dt}{1+t^2}$, and $\cos\theta=\frac{1-t^2}{1+t^2}$. Since the cosine function is even, one has

$$\int_0^{2\pi}\frac{d\theta}{x+\cos\theta}=2\int_0^{\pi}\frac{d\theta}{x+\cos\theta}=2\int_0^{\infty}\frac{2dt}{x(1+t^2)+1-t^2}$$
$$=2\int_0^{\infty}\frac{2dt}{t^2(x-1)+(x+1)}=\frac{4}{x+1}\int_0^{\infty}\frac{dt}{1+(\frac{x-1}{x+1})t^2}. \tag{6.39}$$

Taking $u=\sqrt{\frac{x-1}{x+1}}t$, we obtain

$$\frac{4}{x+1}\int_0^{\infty}\frac{dt}{1+(\frac{x-1}{x+1})t^2}=\frac{2}{x+1}\int_0^{\infty}\frac{2\sqrt{\frac{x+1}{x-1}}du}{u^2+1}=\frac{2}{x+1}\sqrt{\frac{x+1}{x-1}}\int_0^{\infty}\frac{2du}{u^2+1}$$
$$=\frac{2}{x+1}\sqrt{\frac{x+1}{x-1}}\int_0^{\pi}d\theta=\frac{2\pi}{\sqrt{(x-1)(x+1)}}$$
$$=\frac{2\pi}{\sqrt{(\frac{1+y^2}{2y}-1)(\frac{1+y^2}{2y}+1)}}=\frac{4\pi y}{(1-y)(1+y)}. \tag{6.40}$$

Thus,

$$\int_0^{\rho(z)/|z|}\frac{y^3}{2y}\int_0^{2\pi}\frac{d\theta dy}{\frac{1+y^2}{2y}+\cos\theta}=\int_0^{\rho(z)/|z|}\frac{y^2}{2}\frac{4\pi y dy}{(1-y^2)}. \tag{6.41}$$

Let $v=y^2$, then

$$\int_0^{\rho(z)/|z|}\frac{y^2}{2}\frac{4\pi y dy}{(1-y^2)}=\int_0^{(\rho(z)/|z|)^2}\frac{v}{2}\frac{4\pi\sqrt{v}dv}{(1-v)}\frac{dv}{2\sqrt{v}}$$
$$=\pi\int_0^{(\rho(z)/|z|)^2}\frac{v-1+1}{1-v}dv=\pi\int_0^{(\rho(z)/|z|)^2}\left(-1+\frac{1}{1-v}\right)dv$$

$$= \pi \left[-(\frac{\rho(z)}{|z|})^2 - \ln\left(1 - (\frac{\rho(z)}{|z|})^2\right) \right]. \tag{6.42}$$

Hence,

$$MO_{2,1}(f)(z) = \frac{\pi}{\rho(z)} \left[-(\frac{\rho(z)}{|z|})^2 - \ln\left(1 - (\frac{\rho(z)}{|z|})^2\right) \right]^{1/2}. \tag{6.43}$$

Therefore,

$$\begin{aligned} \|f\|^p_{\mathrm{IMO}_1^{p,2,-2/p}} &= \int_{\mathbb{C}} \rho(z)^{-2}\, \mathrm{MO}_{2,1}(f)(z)^p dA(z) \\ &\simeq \int_{\mathbb{C}} \frac{1}{\rho(z)^2} \frac{1}{\rho(z)^p} \left[-(\frac{\rho(z)}{|z|})^2 - \ln\left(1 - (\frac{\rho(z)}{|z|})^2\right) \right]^{p/2} dA(z). \end{aligned} \tag{6.44}$$

Note that taking $x = -(\rho(z)/|z|)^2$, the term in the bracket is $x - \ln(1+x) = x - x + x^2/2 - x^3/3 + \cdots$, and hence the most contribution comes from the term $x^2/2$. Thus,

$$\begin{aligned} \|f\|^p_{\mathrm{IMO}_1^{p,2,-2/p}} &\simeq \int_{\mathbb{C}} \frac{1}{\rho(z)^{p+2}} \frac{\rho(z)^{2p}}{|z|^{2p}} dA(z) = \int_{\mathbb{C}} \frac{1}{\rho(z)^{2-p}} \frac{1}{|z|^{2p}} dA(z) \\ &\geq \int_{|z|\geq R} \frac{1}{|z|^{\beta(2-p)}} \frac{1}{|z|^{2p}} dA(z) \simeq \int_R^\infty \frac{r\,dr}{r^{2p+\beta(2-p)}} \\ &= \int_R^\infty r^{1-2p-\beta(2-p)} dr \simeq r^{2-2p-\beta(2-p)}\Big|_{r=R}^\infty. \end{aligned} \tag{6.45}$$

Note that $2-2p-\beta(2-p) = (2-p)\left(\frac{2(1-p)}{2-p} - \beta\right)$, and since $0 < p \leq 1$, the integral diverges when $\beta \leq \frac{2(1-p)}{2-p}$. So, when $p = 1$, β must be zero. When p is very close to zero, β can get very close to 1, implying that Xia's example is a counterexample for any doubling measure. □

Remark 6.2. One could also hope to modify (6.29) so that it takes into account the growth condition of ρ; see (1.17). However, there are no holomorphic functions that behave like $|z|^c$ at infinity unless c is an integer. Indeed, suppose that f is holomorphic in the complement of a disk centered at the origin, and assume that $\sup_\theta |f(re^{i\theta})| \simeq r^c$ as $r \to \infty$. Then $c \in \mathbb{Z}$. To see this, for such a function f, set $g(z) = z^k f(1/z)$, where $k \leq c$ is an integer. Then g has a removable singularity at the origin since $|g(re^{i\theta})| = \mathcal{O}(r^{k-c})$ as $r \to 0$. So g is bounded at zero, and hence g has a power series $\sum a_k z^k$ near the origin, which implies that $c \in \mathbb{Z}$.

Finally, notice that substituting (6.29) in the proof of Theorem 1.6 by the functions $f(z) = 1/z^n$ for $|z| > 1$ and $f(z) = 0$ elsewhere actually works worse when the integer n is larger than 1.

Proof of Corollary 1.7. We apply Theorem 1.6 to the canonical doubling weights $\phi(z) = |z|^m$ with $m > 0$. Recall that by Lemma 2.5, there is some $R > 0$ such that $\rho(z) \leq |z|^{1-m/2}$ for $|z| \geq R$. Therefore, $\beta_\phi = 1 - m/2$. We can conclude that the Berger-Coburn phenomenon fails for $S_p(F^2_{|z|^m}, F^2_{|z|^m})$ if $1 - m/2 \leq \frac{1-p}{1-p/2}$, which is equivalent to $m \geq \frac{p}{1-p/2}$. In particular, if $m \geq 2$, then the phenomenon fails for all Schatten classes S_p with $0 < p \leq 1$. □

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