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Partial and Full Detection of Source Terms

Liang Zhang, Qing-Guo Wang and Shuang-Hua Yang

***Abstract*—In modern industrial networks, important source terms are monitored during operations. Due to the shortage of sensors or changes of components and connections, all the source terms may not be uniquely detected. This paper is dealt with the problem of partial and full detection of source terms in linear systems. The existence of a partially unique solution for a linear system is analyzed and an algorithm is proposed to identify the maximum set of unique partial solutions. Furthermore, an algorithm is developed to achieve the minimum realization of a unique full solution with additional measurements. The proposed methods are demonstrated through numerical examples and applied to a circuit analysis problem and a source term estimation problem in chemical industrial parks.**

***Index Terms*— source terms detection, system monitoring, linear circuit systems, networks, unique partial solution**

I. INTRODUCTION

In a modern power grid, there could be many energy sources and devices. The grid controller will monitor all its components with real-time measurements and assess their conditions. The electrical model of the system can be built with the circuit analysis [1], [2], and the source terms are monitored. In large and complex networks, identifying all the source terms is required but might be impossible in some situations such as the presence of load uncertainty [3]-[5] and limited availability of measurements [6], [7]. It is noted that it is still possible and meaningful to find some source terms in the network with the current information, and further to determine all the source terms with additional information.

The same detection problem appears in air pollution source term estimation (STE) in chemical industrial parks (CIPs) [8]. STE is to estimate the source parameters, such as source emission rates and source locations, using the real-time concentration measurements, meteorological conditions and other information. However, the number of ambient concentration sensors is insufficient compared with the air pollution sources in CIPs [8]-[11]. Because of this sensor deficiency, the unknown source parameters do not have a unique solution.

It can be seen from the above examples and many others that a common technical problem encountered in scientific or industrial applications involves solving a linear system at some stage and in practice, the system may not have a unique solution [3]-[11]. In general, consider a linear system of m equations in n unknowns,

$$Ax = b \,, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^{n\times 1}$ and $b \in \mathbb{R}^{m\times 1}$. In this paper, we assume that the system is consistent [14], that is, it always has at least one solution and thus meets the following condition,

$$\operatorname{rank}(A) = \operatorname{rank}([A\, b]), \tag{2}$$

where $[Ab]$ is the augmented matrix of the linear system. It is well known that (1) has a unique solution if and only if

$$\operatorname{rank}(A) = \operatorname{rank}([A\, b]) = n \,. \tag{3}$$

In practice, it is possible that a system fails (3) and does not have a unique solution, that is, one cannot determine all variables uniquely. The methods for source term detection under deficient measurements in [8] and [11] reduce the number of variables by clustering the variables into a smaller number of equivalent variables for the unique solution. They have altered the underlying structure of the original system and the solution is not for the original variables. The compressed sensing [12] searches for the sparsest solution to the underdetermined system ($m<n$), and the iteratively re-weighted least squares method [13] can solve this problem. The methods in [9] and [10] augment the number of measurements (equations) at various times and locations to determine all the variables of the system. In engineering practice, it is interesting and useful to consider a general system with no constraint on m, n or x, and uniquely determine the maximal number of the variables with no change on the current system when it has no unique solution for all its variables. In this paper, we term these uniquely determined variables as "unique partial solution", develop the complete theory and algorithm for finding the unique partial solution, and also provide a method to obtain all the system variables with the minimum number of additional measurements. The contributions of this paper are the notion of the unique partial solution of a general linear system, a necessary and sufficient condition for the existence of the unique partial solution, an algorithm for finding the unique partial solution, and a minimum realization of a unique full solution of the system with additional measurements.

The rest of this paper is organized as follows. Section II defines the unique partial solution of a linear system and analyses the unique partial solution for the system. Section III develops a method to determine the full unique solution with minimum additional equations. Section IV applies the new

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theory to the detection of sources in a circuit and an STE problem. Section V concludes this paper.

Notation: e_j donates the *j*-th column of the identity matrix I_n. For any positive integers n and q, $q<n$, let $\boldsymbol{N}=\{1,2,\ldots,n\}$, $K_q=\{k_1,k_2,\ldots,k_q\}\subset\boldsymbol{N}$ and define $\overline{K_q}=\boldsymbol{N}-K_q=\{k_{q+1},k_{q+2},\ldots,k_n\}$ as the complementary set of K_q in $\boldsymbol{N}$. Note that $k_i<k_{i+1}$ and $k_i\neq k_j$ if $i\neq j$. For any vector $x\in\mathbb{R}^n$, let

$$x(K_q)=\begin{bmatrix}x_{k_1}\\x_{k_2}\\\vdots\\x_{k_q}\end{bmatrix},\ x(\overline{K_q})=\begin{bmatrix}x_{k_{q+1}}\\x_{k_{q+2}}\\\vdots\\x_{k_n}\end{bmatrix},\ x(K_q\cup\overline{K_q})=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=x.$$

Partition an $m\times n$ matrix A as

$$A=[a_{\cdot1}\ a_{\cdot2}\ \cdots\ a_{\cdot n}]=[a_{1\cdot}\ a_{2\cdot}\ \cdots\ a_{m\cdot}]^{\mathrm{T}},$$

where $a_{\cdot i}$ is the *i*-th column of A and $a_{j\cdot}$ is the *j*-th row of A. Let $A(K_q)=[a_{\cdot k_1}\ a_{\cdot k_2}\ \cdots\ a_{\cdot k_q}]$, $A(\overline{K_q})=[a_{\cdot k_{q+1}}\ a_{\cdot k_{q+2}}\ \cdots\ a_{\cdot k_n}]$ and it follows that $Ax=A(K_q)x(K_q)+A(\overline{K_q})x(\overline{K_q})$.

II. Unique Partial Solution

The definition of the unique partial solution is given as follows.

Definition 2.1: System (1) under (2) is said to have the unique partial solution if there exists K_{q1} and K_{q2} such that $Ax(K_{q1}\cup\overline{K_{q1}})=b$ and $Ax(K_{q2}\cup\overline{K_{q2}})=b$ imply $x(K_{q1})=x(K_{q2})$.

To determine whether or not the unique partial solution exists, we employ the reduced row echelon form (RREF). A matrix is said to be in the row echelon form [15] if

1) all its nonzero rows are above any rows of all zeros;
2) each leading entry of a row is in a column to the right of the leading entry of the row above it and
3) all its entries in a column below a leading entry are zeros.

A matrix A^{R}, is said to be in the RREF [15] of A if

1) it is in row echelon form;
2) its leading entry in each nonzero row is 1 and
3) each leading 1 is the only nonzero entry in its column.

Lemma 2.1 [14]: A matrix A can be transferred to its reduced row echelon form A^{R}, by left-multiplying a unique nonsingular matrix T: $TA=A^{\mathrm{R}}$, where $T=E_kE_{k-1}\cdots E_1$ and E_1, E_2, …, E_k is a finite sequence of row elementary matrices.

Theorem 2.1: System (1) admits the unique partial solution for a single variable x_j, $j\in\{1,\ldots,n\}$, if and only if there exists a row $a_{i\cdot}^{\mathrm{R}}$ in A^{R}, $i\in\{1,\ldots,m\}$, such that $a_{i\cdot}^{\mathrm{R}}=e_j^{\mathrm{T}}$.

Proof: It follows from Lemma 2.1 that system (1) is row-equivalent to

$$A^{\mathrm{R}}x=b^{\mathrm{R}}, \tag{4}$$

which preserves the solution of the system. Consider one row of (4): $a_{i\cdot}^{\mathrm{R}}x=b_i^{\mathrm{R}}$, which is

$$a_{i1}^{\mathrm{R}}x_1+a_{i2}^{\mathrm{R}}x_2+\cdots+a_{ij}^{\mathrm{R}}x_j+\cdots+a_{in}^{\mathrm{R}}x_n=b_i^{R}. \tag{5}$$

Let $K_q=\{j\}$. If $a_{i\cdot}^{\mathrm{R}}(K_q)=a_{ij}^{\mathrm{R}}=1$ and $a_{i\cdot}^{\mathrm{R}}(\overline{K_q})=0$, that is,

$$a_i^{\mathrm{R}}=[\underbrace{0\ \cdots\ 0}_{\text{zero entry}}\quad\underset{j-\text{th entry}}{1}\quad\underbrace{0\ \cdots\ 0}_{\text{zero entry}}]=e_j^{\mathrm{T}}, \tag{6}$$

then $x_j=b_i^{\mathrm{R}}$ is obtained from (5). Thus, x_j is uniquely determined if (6) holds. On the other hand, if (6) fails, that is $a_{i\cdot}^{\mathrm{R}}\neq e_j^{\mathrm{T}}$ for all $i\in\{1,\ldots,m\}$, then we claim that x_j cannot be solved from (4). It breaks to the following two cases.

- Case 1: $a_{i\cdot}^{\mathrm{R}}\neq e_j^{\mathrm{T}}$ for all $i\in\{1,\ldots,j\}$. There are further two situations.
  i) $a_{ij}^{\mathrm{R}}\neq0$ and there exists at least one element a_{ik}^{R} in $a_{i\cdot}^{\mathrm{R}}$ such that $a_{ik}^{\mathrm{R}}\neq0$, $k\in\{1,\ldots,n\}$ and $k\neq j$. Then (5) becomes $a_{ij}^{\mathrm{R}}x_j+a_{ik}^{\mathrm{R}}x_k+A^{\mathrm{R}}(\overline{K_q})x(\overline{K_q})=b_i^{\mathrm{R}}$, where $K_q=\{j,k\}$. The above single equation has two unknowns with nonzero a_{ij}^{R} and a_{ik}^{R}, and it cannot uniquely determine x_j.
  ii) $a_{ij}^{\mathrm{R}}=0$. Then (5) contains no term with x_j and it cannot determine x_j.
- Case 2: $a_{i\cdot}^{\mathrm{R}}\neq e_j^{\mathrm{T}}$ for all $i\in\{j+1,\ldots,m\}$. The specific format of RREF indicates $a_{ij}^{\mathrm{R}}=0$ for all $i\in\{j+1,\ldots,m\}$. Then (5) contains no term with x_j and it cannot determine x_j.

The proof is completed. □

Theorem 2.2: System (1) admits the unique partial solution for a single variable x_j, $j\in\{1,\ldots,n\}$, that is, $K_q=\{j\}$, if and only if

$$\operatorname{rank}\left(A(\overline{K_q})\right)=\operatorname{rank}(A)-1. \tag{7}$$

Proof: By Theorem 2.1, x_j for $K_q=\{j\}$ is uniquely determined in the system if and only if there exists a row in A^{R} such that $a_{i\cdot}^{\mathrm{R}}=e_j^{\mathrm{T}}$. This condition is equivalent to

$$\operatorname{rank}\left(A^{\mathrm{R}}(\overline{K_q})\right)=\operatorname{rank}(A^{\mathrm{R}})-1, \tag{8}$$

since $A^{\mathrm{R}}(\overline{K_q})$ is formed by deleting the *j*-th column of A^{R} and its independent columns are exactly one less than those of A^{R}. Note that $A^{\mathrm{R}}(\overline{K_q})=A^{\mathrm{R}}M$ and $M=[e_1,e_2,\ldots,e_{j-1},e_{j+1},\ldots,e_n]$. By Lemma 2.1, there exists a unique nonsingular matrix T such that $TA=A^{\mathrm{R}}$. Then it follows that $TAM=A^{\mathrm{R}}M$ or $TA(\overline{K_q})=A^{\mathrm{R}}(\overline{K_q})$. Since the elementary row operations do not change the rank of the matrix, we have

$$\operatorname{rank}(A^{\mathrm{R}})=\operatorname{rank}(A), \tag{9}$$

and

$$\operatorname{rank}\left(A^{\mathrm{R}}(\overline{K_q})\right)=\operatorname{rank}\left(A(\overline{K_q})\right). \tag{10}$$

Equations (8)-(10) yield (7). The proof is completed. □

By Theorem 2.2, we can check the existence of the unique partial solution and find which variables can be identified in the system. Algorithm 1 shows how this is done.

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Algorithm 1 Find Unique Maximum Partial Solution

Input A and b ; let $K_q = \varnothing$;
for $j = 1$ to n, $K^j = \{j\}$
  if $\operatorname{rank}\left(A\left(\overline{K^j}\right)\right) = \operatorname{rank}(A) - 1$ **then** $K_q = K_q \cup \{j\}$
Output K_q

We use an example to illustrate Algorithm 1.

Example 2.1: Consider a 4×5 linear system with

$$A = \begin{bmatrix} -1 & -2 & 5 & -2 & 3 \\ -2 & -4 & 3 & -3 & 6 \\ 3 & 6 & -4 & 5 & -9 \\ -3 & -6 & 8 & -5 & 9 \end{bmatrix},\ b = \begin{bmatrix} 5 \\ -3 \\ 6 \\ 2 \end{bmatrix}. \tag{11}$$

Note that $n = 5$ and $\operatorname{rank}(A) = 3$. The iterative process of Algorithm 1 is shown in Table I.

TABLE I. EXAMPLE 2.1 WITH ALGORITHM 1

j	K^j	$\operatorname{rank}\left(A\left(\overline{K^j}\right)\right)$	$\operatorname{rank}\left(A\left(\overline{K^j}\right)\right) = \operatorname{rank}(A) - 1$	K_q
1	{1}	3	False	$\varnothing$
2	{2}	3	False	$\varnothing$
3	{3}	2	True	{3}
4	{4}	2	True	{3,4}
5	{5}	3	False	{3,4}

Finally, output $K_q = \{3,4\}$, and $[x_3\ \ x_4]^{\mathrm{T}}$ can be uniquely determined in the system. The algorithm ends.

Theorem 2.2 can be extended to the multivariable case as follows.

Theorem 2.3: System (1) admits the unique partial solution for $K_q = \{k_1, k_2, ..., k_q\}$ if and only if

$$\operatorname{rank}\left(A\left(\overline{K_q}\right)\right) = \operatorname{rank}(A) - q\ . \tag{12}$$

Proof: System (1) admits the unique partial solution for $K_q = \{k_1, k_2, \dots, k_q\}$, that is, for $l = 1, 2, ..., q$, x_{k_l} is uniquely determined in the system. By Theorem 2.1, this is equivalent to there exists some row $a_{i\cdot}^{\mathrm{R}}$ in A^{R}, such that $a_{i\cdot}^{\mathrm{R}} = e_{k_l}^{\mathrm{T}}$, or

$$\operatorname{rank}\left(A^{\mathrm{R}}\left(\overline{K_q}\right)\right) = \operatorname{rank}\left(A^{\mathrm{R}}\right) - q\ . \tag{13}$$

This is because $A^{\mathrm{R}}\left(\overline{K_q}\right)$ is formed by deleting the columns $a_{\cdot k_1}^{\mathrm{R}}$, $a_{\cdot k_2}^{\mathrm{R}}$, …, $a_{\cdot k_q}^{\mathrm{R}}$ of A^{R} and these q columns are mutually independent. Thus, the independent columns of $A^{\mathrm{R}}\left(\overline{K_q}\right)$ are exactly q less than those of A^{R}. Note that $A^{\mathrm{R}}\left(\overline{K_q}\right) = A^{\mathrm{R}} M$, where $M = I_n\left(\overline{K_q}\right)$. Likewise, (9) and (10) still hold in this multivariable case. Then equations (9), (10) and (13) yield (12). The proof is completed. □

To illustrate Theorem 2.3, consider again the system in (11). Let $K_q = \{3,4\}$ with $q = 2$. Then one sees

$$A\left(\overline{K_q}\right) = \left[a_{\cdot 1}\ \ a_{\cdot 2}\ \ a_{\cdot 5}\right] = \begin{bmatrix} -1 & -2 & 3 \\ -2 & -4 & 6 \\ 3 & 6 & -9 \\ -3 & -6 & 9 \end{bmatrix},$$

which has three columns proportional to each other, indicating

$$\operatorname{rank}\left(A\left(\overline{K_q}\right)\right) = 1 = \operatorname{rank}(A) - q\ .$$

This is the same solution as the one obtained from Theorem 2.2.

III. MINIMUM REALIZATION OF THE UNIQUE FULL SOLUTION

If the system (1) admits the unique partial solution but no unique full solution, some variables cannot be determined. Additional measurements should be made to find other variables. But the new measurement equations should create new information in the sense that they can be used to find all the variables uniquely, and the cost for doing so should be minimized. This leads to the problem of the minimum realization of the unique full solution of a given system.

We consider adding a row to (1) at a time to find more variables. Suppose that a new measurement on x is made, which gives a new equation $a_{m+1\cdot} x = b_{m+1}$. Stack it with (1) as $A_1 x = B_1$, where

$$A_1 = \begin{bmatrix} A \\ a_{m+1\cdot} \end{bmatrix},\ B_1 = \begin{bmatrix} b \\ b_{m+1} \end{bmatrix}.$$

We give an algorithm to construct additional equations for the unique full solution of a system. Specifically, we search from $j = 1$. If x_j is uniquely determined in the current system, $Ax = b$, according to Theorem 2.2, proceed to $j+1$; otherwise augment A with e_j^{T} as the last row and get a new measurement b_{m+1}, so that the last equation of the newly formed system has the unique solution for x_j, and there holds

$$\operatorname{rank}\begin{bmatrix} A \\ e_j^{\mathrm{T}} \end{bmatrix} = \operatorname{rank}(A) + 1\ .$$

In the end of the search, with $n - \operatorname{rank}(A)$ additional equations, the final system will have full rank and a unique full solution. Algorithm 2 shows the steps of getting the unique full solution.

Algorithm 2 Find Unique Full Solution

Input A and b
Set $l = 0$, $A_0 = A$, $B_0 = b$
for $j = 1$ to n, $K^j = \{j\}$
  if $\operatorname{rank}(A_l(\overline{K^j})) = \operatorname{rank}(A_l)$ **then**
    $l = l + 1$, $A_l = \begin{bmatrix} A_{l-1} \\ e_j^{\mathrm{T}} \end{bmatrix}$, $B_l = \begin{bmatrix} B_{l-1} \\ b_{m+l} \end{bmatrix}$
    if $\operatorname{rank}(A_l) = n$ **then**
      break
Solve $A_l x = B_l$
Output x

To illustrate Algorithm 2, consider again the system in (11). Let $l = 0$, $A_0 = A$ and $B_0 = b$. Note that $m = 4$, $n = 5$ and $\operatorname{rank}(A) = 3$. The procedure of iterations is shown in Table II.

TABLE II. EXAMPLE 2.1 WITH ALGORITHM 2

j	K^j	$\operatorname{rank}\left(A_l\left(\overline{K^j}\right)\right) = \operatorname{rank}(A_l)$	l	A_l	B_l	$\operatorname{rank}(A_l) = n$
1	{1}	True	1	$\left[A_0\ e_1^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_0\ b_5\right]^{\mathrm{T}}$	False
2	{2}	True	2	$\left[A_1\ e_2^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_1\ b_6\right]^{\mathrm{T}}$	True

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Because $\text{rank}(A_2)=5=n$, then one comes out of the loop. Based on the above analysis, two additional measurements b_5 and b_6 are required to determine the other sources. For example, let $b_5=b_6=1$. Finally, by solving the full-rank system $A_2x=B_2$, the unique full solution x is obtained as $x=[1\ 1\ 2\ 1\ 0]^{\text{T}}$.

IV. Applications

In this section, two reduced-scale cases of real-life industrial networks are presented due to limited space.

Example 4.1: Consider the circuit in Fig. 1. There are 10 current sources $i_1 \sim i_{10}$ to be determined in this circuit. Let $x=[i_1\ \ i_2\ \ \ldots\ \ i_{10}]^{\text{T}}$. $R_1 \sim R_5$ are the resistors with resistance values 4Ω, 3Ω, 1Ω, 2Ω and 4Ω. The measurements are the voltage across the resistors: $[V_1\ \ V_2\ \ \ldots\ \ V_5]=[24\ \ 27\ \ 15\ \ -12\ \ 16]$ (V). The parameter settings of the circuit are taken within actual ranges of the components (resistors and sources) with possibly simple values solely for illustration of the proposed theory and algorithm.

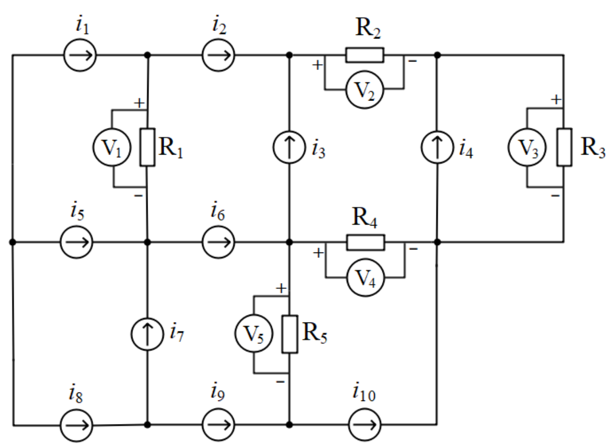

Fig. 1. Circuit with 10 current sources $i_1 \sim i_{10}$.

Apply the Kirchhoff's Current Law (KCL) to each node, a system in (1) is obtained with

$$A=\begin{bmatrix} 4 & -4 & & & & & & & & \\ & 3 & 3 & & & & & & & \\ & & & 3 & & & & & & \\ -1 & & & & -1 & & & -1 & & \\ & & & & 4 & -4 & 4 & & & \\ & & -4 & & & 4 & & & & \\ & & & -2 & & & & & & 2 \\ & & & & & & -1 & 1 & -1 & \\ & & & & & & & & 4 & -4 \end{bmatrix},\ b=\begin{bmatrix} 24 \\ 27 \\ 18 \\ 0 \\ -24 \\ -8 \\ -18 \\ 0 \\ -16 \end{bmatrix}.$$

where the blank parts of the above matrix are zero. Note that $m=9$, $n=10$ and $\text{rank}(A)=8$. We first use Algorithm 1 to find which current source can be uniquely determined. The iterative steps of the algorithm are shown in Table III. Finally, output $K_q=\{4,9,10\}$ which means i_4, i_9 and i_{10} are uniquely determined. Then we use Algorithm 2 to determine all the sources. Let $l=0$, $A_0=A$ and $B_0=b$. The iterative process is illustrated in Table IV.

Table III. Example 4.1 with Algorithm 1

j	K^j	$\text{rank}\left(A\left(\overline{K^j}\right)\right)$	$\text{rank}\left(A\left(\overline{K^j}\right)\right)=\text{rank}(A)-1$	K_q
1	{1}	8	False	∅
2	{2}	8	False	∅
3	{3}	8	False	∅
4	{4}	7	True	{4}
5	{5}	8	False	{4}
⋮	⋮	8	False	{4}
8	{8}	8	False	{4}
9	{9}	7	True	{4,9}
10	{10}	7	True	{4,9,10}

Table IV. Example 4.1 with Algorithm 2

j	K^j	$\text{rank}\left(A_l\left(\overline{K^j}\right)\right)=\text{rank}(A_l)$	l	A_l	B_l	$\text{rank}(A_l)=n$
1	{1}	True	1	$\left[A_0\ e_1^{\text{T}}\right]^{\text{T}}$	$\left[B_0\ b_{10}\right]^{\text{T}}$	False
2	{2}	False	1	$\left[A_0\ e_1^{\text{T}}\right]^{\text{T}}$	$\left[B_0\ b_{10}\right]^{\text{T}}$	False
⋮	⋮	False	1	$\left[A_0\ e_1^{\text{T}}\right]^{\text{T}}$	$\left[B_0\ b_{10}\right]^{\text{T}}$	False
5	{5}	True	2	$\left[A_1\ e_5^{\text{T}}\right]^{\text{T}}$	$\left[B_1\ b_{11}\right]^{\text{T}}$	True

Because $\text{rank}(A_2)=10=n$, then one comes out of the loop. Based on the above analysis, two additional measurements b_{10} and b_{11} are required to determine the other current sources. For example, let the new measurements $[b_{10}\ \ b_{11}]=[8\ \ -5]$. By solving the full-rank system $A_2x=B_2$, we identify all the current sources in the circuit:

$$x=[i_1\ \ i_2\ \ \ldots\ \ i_{10}]^{\text{T}}=[8\ \ 2\ \ 7\ \ 6\ \ -5\ \ 5\ \ 4\ \ -3\ \ -7\ \ -3]^{\text{T}}.$$

Example 4.2: Consider an STE problem in a CIP. There are 10 sources ($S_1 \sim S_{10}$) with emission rate $x=[x_1\ \ x_2\ \ \ldots\ \ x_{10}]^{\text{T}}$ (mg/s) monitored by 6 receptors ($M_1 \sim M_6$) (shown in Fig. 2). The wind speed is 1m/s and the direction is parallel to the positive X-axis.

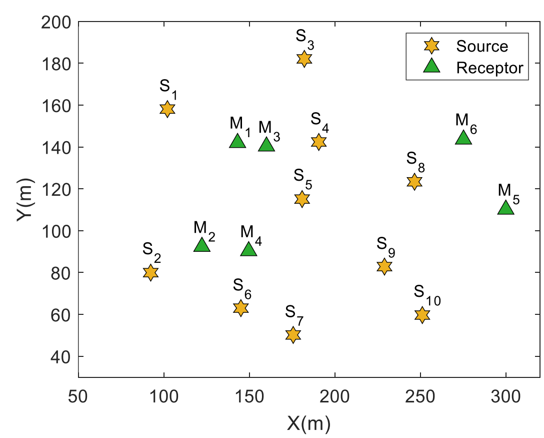

Fig. 2. Locations of 10 sources and 6 receptors.

The model in the form of (1) is obtained [9] with

$$A=\begin{bmatrix} 86 & 96 & 45 & 222 & 292 & 113 & 83 & 799 & 343 & 7 \\ 144 & 72 & 186 & 539 & 284 & 36 & 5 & 182 & 0 & 0 \\ 596 & 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 698 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 832 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 854 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ b=\begin{bmatrix} 6659 \\ 4899 \\ 1963 \\ 2260 \\ 1951 \\ 2003 \end{bmatrix}\times 10^3,$$

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where b is the measurements (ng/m^3). Note that $m = 6$, $n = 10$ and $\text{rank}(A) = 4$. We first use Algorithm 1 to check which source can be identified and the execution process is illustrated in Table V. Finally, output $K_q = \{1, 2\}$, and the emission sources x_1 and x_2 are uniquely determined. Next we use Algorithm 2 to determine all the source terms. Let $l = 0$, $A_0 = A$ and $B_0 = b$. The iterative steps are shown in Table VI.

TABLE V. EXAMPLE 4.2 WITH ALGORITHM 1

j	K^j	$\text{rank}\left(A\left(\overline{K^j}\right)\right)$	$\text{rank}\left(A\left(\overline{K^j}\right)\right) = \text{rank}(A) - 1$	K_q
1	{1}	3	True	{1}
2	{2}	3	True	{1, 2}
3	{3}	4	False	{1, 2}
⋮	⋮	4	False	{1, 2}
10	{10}	4	False	{1, 2}

TABLE VI. EXAMPLE 4.2 WITH ALGORITHM 2

j	K^j	$\text{rank}\left(A_l\left(\overline{K^j}\right)\right) = \text{rank}(A_l)$	l	A_l	B_l	$\text{rank}(A_l) = n$
1	{1}	False	0	A_0	B_0	False
2	{2}	False	0	A_0	B_0	False
3	{3}	True	1	$\left[A_0\ e_3^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_0\ b_7\right]^{\mathrm{T}}$	False
4	{4}	True	2	$\left[A_1\ e_4^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_1\ b_8\right]^{\mathrm{T}}$	False
5	{5}	True	3	$\left[A_2\ e_5^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_2\ b_9\right]^{\mathrm{T}}$	False
6	{6}	True	4	$\left[A_3\ e_6^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_3\ b_{10}\right]^{\mathrm{T}}$	False
7	{7}	True	5	$\left[A_4\ e_7^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_4\ b_{11}\right]^{\mathrm{T}}$	False
8	{8}	False	5	$\left[A_4\ e_7^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_4\ b_{11}\right]^{\mathrm{T}}$	False
9	{9}	True	6	$\left[A_5\ e_9^{\mathrm{T}}\right]^{\mathrm{T}}$	$\left[B_5\ b_{12}\right]^{\mathrm{T}}$	True

Because $\text{rank}(A_6) = 10 = n$, then one exits the loop. Based on the above analysis, six additional measurements b_7~b_{12} are required to determine the other sources. For example, let

$$\left[b_7\ b_8\ \cdots\ b_{12}\right] = \left[4132\ 3901\ 2323\ 2761\ 2578\ 3374\right].$$

By solving the full-rank system $A_6 x = B_6$, all the source terms are obtained as

$$x = \left[3234\ 2345\ 4132\ 3901\ 2323\ 2761\ 2578\ 3415\ 3374\ 2004\right]^{\mathrm{T}}.$$

V. CONCLUSION

In this paper, we have addressed partial and full detection of the source terms in linear systems. We have analyzed the existence of the partially unique solution for a linear system and proposed an algorithm to identify the maximum set of the unique partial solution. Furthermore, we have developed an algorithm to achieve the minimum realization of a unique full solution with additional measurements. The proposed methods have been demonstrated through examples and applied to a circuit analysis problem and an STE problem.

REFERENCES

[1] K. C. A. Smith and R. E. Alley, *Electrical Circuit: An Introduction*. Cambridge, U.K.: Cambridge Univ. Press, 1992.

[2] C. K. Tse, *Linear Circuit Analysis*, London, U.K.: Addison Wesley, 1998.

[3] C. Mu, T. Ding, Y. Sun, S. Xieliu, X. He and L. Li, "Lifting Model based Interval Analysis for DC Power Flow in Presence of Uncertain Network Reactance and Power Injections," *IEEE Trans. Circuits Syst. II: Exp. Briefs* vol. 70, no. 7, pp. 2500-2504, July 2023.

[4] G. Liu, C. Yuan, X. Chen, J. Wu, R. Dai and Z. Wang, "A High-Performance Energy Management System Based on Evolving Graph," *IEEE Trans. Circuits Syst. II: Exp. Briefs*, vol. 67, no. 2, pp. 350-354, Feb. 2020.

[5] T. Kobaku, R. Jeyasenthil, S. Sahoo, R. Ramchand and T. Dragicevic, "Quantitative Feedback Design-Based Robust PID Control of Voltage Mode Controlled DC-DC Boost Converter," *IEEE Trans. Circuits Syst. II: Exp. Briefs*, vol. 68, no. 1, pp. 286-290, Jan. 2021.

[6] S. Azizi, M. Sanaye-Pasand and M. Paolone, "Locating Faults on Untransposed, Meshed Transmission Networks Using a Limited Number of Synchrophasor Measurements", *IEEE Trans. Power Appar. Syst.*, vol.31, no.6, pp.4462-4472, Nov. 2016.

[7] Z. Wang, Z. Liu, Y. Dai and K. Cai, "Recursive Estimation for System with Random Transmission Loss and Censored Measurement," *IEEE Trans. Circuits Syst. II: Exp. Briefs*, vol. 70, no. 6, pp. 2281-2285, June 2023.

[8] K. Cheng, X. Zhao, W. Zhou, Y. Cao, S.-H. Yang and J. Chen, "Source term estimation with deficient sensors: Traceability and an equivalent source approach," *Process Saf. Environ. Prot.*, vol. 152, pp. 131-139, Aug. 2021.

[9] X. Zhao, K. Cheng, W. Zhou, Y. Cao, S.-H. Yang and J. Chen, "Source term estimation with deficient sensors: A temporal augment approach," *Process Saf. Environ. Prot.*, vol. 157, pp. 131-139, Jan. 2022.

[10] W. Zhou, X. Zhao, K. Cheng, Y. Cao, S.-H. Yang and J. Chen, "Source term estimation with deficient sensors: Error analysis and mobile station route design," *Process Saf. Environ. Prot.*, vol. 154, pp. 97-103, Oct. 2021.

[11] K. Cheng, X. Zhao, W. Zhou, Y. Cao and S.-H. Yang, "Excess Emission Sources Identification via Sparse Monitoring Networks," *IEEE Trans. Ind. Inf.*, Jan. 2023. doi: 10.1109/TII.2023.3240560.

[12] D. L. Donoho, Y. Tsaig, I. Drori and J. L. Starck, "Sparse Solution of Underdetermined Systems of Linear Equations by Stagewise Orthogonal Matching Pursuit," *IEEE Trans. Inf. Theory*, vol. 58, no. 2, pp. 1094-1121, Feb. 2012.

[13] D. Ba, B. Babadi, P. L. Purdon and E. N. Brown, "Convergence and Stability of Iteratively Re-weighted Least Squares Algorithms," *IEEE Trans. Signal Process.*, vol. 62, no. 1, pp. 183-195, Jan. 2014.

[14] S. J. Leon, "Matrices and Systems of Equations," in *Linear Algebra with Applications*, 8th ed. New York, NY, U.S.: Pearson, 2009, ch. 1, sec. 5, pp. 61.

[15] D. C. Lay, S. R. Lay and J. J. Mcdonald, "Linear Equations in Linear Algebra," in *Linear algebra and its applications*. 5th ed. London, U.K.: Pearson, 2015, ch. 1, sec. 2, pp. 13.