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Generalized Volterra type integral operators on large Bergman spaces

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ABSTRACT

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ and g analytic in $\mathbb{D}$. We characterize boundedness and compactness of generalized Volterra type integral operators

$$GI_{(\phi,g)}f(z) = \int_0^z f'(\phi(\xi))\, g(\xi)\, d\xi$$

and

$$GV_{(\phi,g)}f(z) = \int_0^z f(\phi(\xi))\, g(\xi)\, d\xi,$$

acting between large Bergman spaces A^p_ω and A^q_ω for $0 < p, q \leq \infty$. To prove our characterizations, which involve Berezin type integral transforms, we use the Littlewood-Paley formula of Constantin and Peláez and establish corresponding embedding theorems, which are also of independent interest.

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When $\phi(z) = z$, our results for $GV_{(\phi,g)}$ complement the descriptions of Pau and Peláez.

1. Introduction and main results

For $0 < p < \infty$ and a positive function $\omega \in L^1(\mathbb{D}, dA)$, the weighted Bergman spaces A^p_ω and A^∞_ω consist of all analytic functions defined on the unit disk $\mathbb{D}$ for which

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \, \omega(z)^{p/2} \, dA(z) < \infty$$

and

$$\|f\|_{A^\infty_\omega} = \sup_{z\in\mathbb{D}} |f(z)| \, \omega(z)^{1/2} < \infty,$$

respectively, where dA is the normalized area measure on $\mathbb{D}$.

In this paper, we study generalized Volterra type integral operators between weighted Bergman spaces for a certain class $\mathcal{W}$ of radial rapidly decreasing weights. The class $\mathcal{W}$, considered previously in [4] and [15], consists of the radial decreasing weights of the form $\omega(z) = e^{-2\varphi(z)}$, where $\varphi \in C^2(\mathbb{D})$ is a radial function such that $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$ for some radial positive function $\tau(z)$ that decreases to 0 as $|z| \to 1^-$ and satisfies $\lim_{r\to 1^-} \tau'(r) = 0$. Here Δ denotes the standard Laplace operator. Furthermore, we assume that there either exists a constant $C > 0$ such that $\tau(r)(1-r)^{-C}$ increases for r close to 1 or

$$\lim_{r\to 1^-} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

See Section 7 of [15] for examples of weights in $\mathcal{W}$, such as the following exponential type weight

$$\omega_{\gamma,\alpha}(z) = (1-|z|)^\gamma \exp\left(\frac{-b}{(1-|z|)^\alpha}\right), \quad \gamma \geq 0, \alpha > 0, b > 0.$$

For the weights ω in $\mathcal{W}$, the point evaluations $L_z : f \longmapsto f(z)$ are bounded linear functionals on A^2_ω for each $z \in \mathbb{D}$, and so A^2_ω is a reproducing kernel Hilbert space; that is, for each $z \in \mathbb{D}$, there are functions $K_z \in A^2_\omega$ with $\|L_z\| = \|K_z\|_{A^2_\omega}$ such that $L_z f = f(z) = \langle f, K_z\rangle_\omega$, where

$$\langle f, g\rangle_\omega = \int_{\mathbb{D}} f(z) \, \overline{g(z)} \, \omega(z) \, dA(z).$$

The function K_z is called the reproducing kernel for the Bergman space A^2_ω and has the property that $K_z(\xi) = \overline{K_\xi(z)}$. The Bergman spaces with exponential type weights have attracted considerable attention in recent years because of novel techniques different from those used for standard Bergman spaces; see, e.g., [2,5] and the references therein. Various estimates for the reproducing kernel play an important role in our work and we discuss them further in Section 2.1.

Let ϕ and g be analytic self-maps of $\mathbb{D}$. The generalized Volterra type integral operators $GI_{(\phi,g)}$ and $GV_{(\phi,g)}$ induced by the pair of symbols (ϕ, g) are defined by

$$GI_{(\phi,g)}f(z) = \int_0^z f'(\phi(\xi))\, g(\xi) d\xi \quad \text{and} \quad GV_{(\phi,g)}f(z) = \int_0^z f(\phi(\xi))\, g(\xi) d\xi, \tag{1.1}$$

where $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. When $g = \phi'$, the operator $GI_{(\phi,\phi')}$ is the composition operator C_ϕ up to a certain constant—these operators acting between different large Bergman spaces were recently studied in [1]. As another special case, when $\phi(\xi) = \xi$, we obtain the Volterra integral operator

$$V_g f(z) := GV_{(\phi,g')}f(z) = \int_0^z f(\xi)\, g'(\xi) d\xi, \tag{1.2}$$

and its companion integral operator

$$J_g f(z) := GI_{(\phi,g)}f(z) = \int_0^z f'(\xi)\, g(\xi) d\xi. \tag{1.3}$$

Previously Pau and Peláez characterized boundedness and compactness of $V_g : A^p_\omega \to A^q_\omega$ in [15] when $0 < p, q < \infty$. Via (1.2) and (1.3), our characterizations extend the previous results to the full range $0 < p, q \leq \infty$ and to all weights in $\mathcal{W}$, and also deal with the companion operator J_g for the first time. The generalized Volterra type integral operators $GI_{(\phi,g)}$ and $GV_{(\phi,g)}$ were previously studied by Mengestie [11–13] in standard Fock spaces and by Li [6] in standard Bergman spaces and Bloch type spaces.

1.1. *Main results*

In this paper we study boundedness and compactness of the generalized Volterra type integral operators $GI_{(\phi,g)}$ and $GV_{(\phi,v)}$. Our results on Schatten class properties, compact differences, and the essential norm of these operators will be published elsewhere.

For $0 < p, q < \infty$, our characterizations for boundedness and compactness are given in terms of the integral transform

$$GB^{\phi}_{n,p,q}(g)(z) = \int_{\mathbb{D}} |k_{p,z}(\phi(\xi))|^q \, \frac{(1 + \varphi'(\phi(\xi))^{nq}}{(1 + \varphi'(\xi))^q} \, |g(\xi)|^q \, \omega(\xi)^{q/2} \, dA(\xi), \quad z \in \mathbb{D},$$

where $n = 0, 1$ and $k_{p,z}$ is the normalized reproducing kernel of A^p_ω.

Theorem 1.1. *Let $\omega \in \mathcal{W}$, $\phi : \mathbb{D} \to \mathbb{D}$ be analytic, and $g \in H(\mathbb{D})$.*

(A) *For $0 < p \leq q < \infty$, the operator $GI_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded if and only if*

$$GB^\phi_{1,p,q}(g) \in L^\infty(\mathbb{D}, dA),$$

and compact if and only if $\lim_{|z|\to 1^-} GB^\phi_{1,p,q}(g)(z) = 0$.

(B) *For $0 < p < \infty$, $GI_{(\phi,g)} : A^p_\omega \to A^\infty_\omega$ is bounded if and only if*

$$MI_{g,\phi,\omega}(z) := |g(z)| \, \frac{(1 + \varphi'(\phi(z))}{(1 + \varphi'(z))} \frac{\omega(z)^{1/2}}{\omega(\phi(z))^{1/2}} \Delta\varphi(\phi(z))^{1/p} \in L^\infty(\mathbb{D}, dA), \tag{1.4}$$

and compact if and only if $\lim_{|\phi(z)|\to 1^-} MI_{g,\phi,\omega}(z) = 0$.

(C) *The operator $GI_{(\phi,g)} : A^\infty_\omega \to A^\infty_\omega$ is bounded if and only if*

$$NI_{g,\phi,\omega}(z) := |g(z)| \, \frac{(1 + \varphi'(\phi(z))}{(1 + \varphi'(z))} \frac{\omega(z)^{1/2}}{\omega(\phi(z))^{1/2}} \in L^\infty(\mathbb{D}, dA), \tag{1.5}$$

and compact if and only if $\lim_{|\phi(z)|\to 1^-} NI_{g,\phi,\omega}(z) = 0$.

(D) *For $0 < q < p \leq \infty$, both boundedness and compactness of $GI_{(\phi,g)} : A^p_\omega \to A^q_\omega$ are equivalent to the condition*

$$GB^\phi_{1,p,q}(g) \in L^s(\mathbb{D}, d\lambda),$$

where $\lambda(z) = dA(z)/\tau(z)^2$ and $s = p/(p - q)$ if $p < \infty$ and $s = 1$ if $p = \infty$.

Theorem 1.2. *Let $\omega \in \mathcal{W}$, $\phi : \mathbb{D} \to \mathbb{D}$ be analytic, and $g \in H(\mathbb{D})$.*

(A) *For $0 < p \leq q < \infty$, $GV_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded if and only if $GB^\phi_{0,p,q}(g) \in L^\infty(\mathbb{D}, dA)$, and compact if and only if $\lim_{|z|\to 1^-} GB^\phi_{0,p,q}(g) = 0$.*

(B) *For $0 < p < \infty$, the operator $GV_{(\phi,g)} : A^p_\omega \to A^\infty_\omega$ is bounded if and only if*

$$MV_{g,\phi,\omega}(z) := \frac{|g(z)|}{(1 + \varphi'(z))} \frac{\omega(z)^{1/2}}{\omega(\phi(z))^{1/2}} \Delta\varphi(\phi(z))^{1/p} \in L^\infty(\mathbb{D}, dA), \tag{1.6}$$

and compact if and only if $\lim_{|\phi(z)|\to 1} MV_{g,\phi,\omega}(z) = 0$.

(C) *The operator $GV_{(\phi,g)} : A^\infty_\omega \to A^\infty_\omega$ is bounded if and only if*

$$NV_{g,\phi,\omega}(z) := \frac{|g(z)|}{(1 + \varphi'(z))} \frac{\omega(z)^{1/2}}{\omega(\phi(z))^{1/2}} \in L^\infty(\mathbb{D}, dA), \tag{1.7}$$

and compact if and only if $\lim_{|\phi(z)|\to 1} NV_{g,\phi,\omega}(z) = 0$.

(D) *For $0 < q < p \leq \infty$, both boundedness and compactness of $GV_{(\phi,g)} : A^p_\omega \to A^q_\omega$ are equivalent to the condition*

$$GB^{\phi}_{0,p,q}(g) \in L^r(\mathbb{D}, d\lambda),$$

where $r = p/(p-q)$ when $p < \infty$ and $r = 1$ when $p = \infty$.

We also prove the following simpler necessary conditions for boundedness and compactness.

Proposition 1.3. *Let $\omega \in \mathcal{W}$, $\phi : \mathbb{D} \to \mathbb{D}$ be analytic, and $g \in H(\mathbb{D})$.*

(A) *If $0 < p, q < \infty$ and $GI_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded, then*

$$z \mapsto |g(z)| \frac{\tau(z)^{2/q}}{\tau(\phi(z))^{2/p}} \frac{(1+\varphi'(\phi(z))}{(1+\varphi'(z))} \frac{\omega(z)^{1/2}}{\omega(\phi(z))^{1/2}} \in L^\infty(\mathbb{D}, dA); \tag{1.8}$$

and if $GI_{(\phi,q)}$ is compact, then the function in (1.8) vanishes as $|z| \to 1$.

(B) *If $0 < p \le q < \infty$ and $GV_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded, then*

$$z \mapsto \frac{\tau(z)^{2/q}}{\tau(\phi(z))^{2/p}} \frac{|g(z)|}{(1+\varphi'(z))} \frac{\omega(z)^{1/2}}{\omega(\phi(z))^{1/2}} \in L^\infty(\mathbb{D}, dA); \tag{1.9}$$

and if $VG_{(\phi,g)}$ is compact, then the function in (1.9) vanishes as $|\phi(z)| \to 1$.

As a consequence of the two main theorems, when $\phi(z) = z$, we obtain characterizations for boundedness and compactness of V_g and its companion operator J_g. The results for J_g are new while the descriptions for V_g had been partially obtained before as explained in the following remark.

Remark 1.4. Notice that (C) and (D) of Corollary 1.5 are analogous to the descriptions given in Theorem 3 of Constantin and Peláez [3] when V_g is acting between weighted Fock spaces, but the two cases require different methods due to fundamental differences between the two types of spaces. Further, Corollary 1.6 implies Theorem 2 of Pau and Peláez [15], that is, we show that the their conditions are equivalent of those in Theorem 1.2 when $\phi(z) = z$.

Corollary 1.5. *Let $\omega \in \mathcal{W}$ and $g \in H(\mathbb{D})$.*

(A) *For $0 < p < q \le \infty$, $J_g : A^p_\omega \to A^q_\omega$ is bounded if and only if $g = 0$.*

(B) *For $p > q$, $J_g : A^p_\omega \to A^q_\omega$ is compact if and only if $g \in L^s(\mathbb{D}, dA)$, where $s = pq/(p-q)$.*

(C) *For $0 < p \le q \le \infty$, $V_g : A^p_\omega \to A^q_\omega$ is bounded if and only if*

$$z \mapsto \frac{|g'(z)|}{(1+\varphi'(z))} \Delta\varphi(z)^{\frac{1}{p}-\frac{1}{q}} \in L^\infty(\mathbb{D}, dA), \tag{1.10}$$

and $V_g : A^p_\omega \to A^q_\omega$ is compact if the function in (1.10) vanishes as $|z| \to 1$.

(D) *For* $0 < q < p < \infty$, $V_g : A^p_\omega \to A^q_\omega$ *is bounded if and only if*

$$\frac{|g'(z)|}{(1+\varphi'(z))} \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA). \tag{1.11}$$

In the next corollary, we consider the weighted Bergman space $A^p(\omega) = A^p_{\omega^{2/p}}$, that is,

$$A^p(\omega) = \left\{ f \in H(\mathbb{D}) : \|f\|^p_{A^p(w)} = \int_{\mathbb{D}} |f(z)|^p \, \omega(z)\, dA(z) < \infty \right\},$$

where the weight $\omega \in \mathcal{W}$ satisfies the condition

$$\Delta\phi(z) \asymp ((1-|z|)^t \psi_\omega(z))^{-1}, \; z \in \mathbb{D}, \text{ for some } t \geq 1.$$

In particular, we obtain the conditions of Theorem 2 in [15] for boundedness and compactness of the operator $V_g : A^p(\omega) \to A^q(\omega)$.

Corollary 1.6. *Let* $0 < p, q < \infty$, $\omega \in \mathcal{W}$, *and* $g \in H(\mathbb{D})$.

(I) *For* $p = q$, *we have the following statements*
  (a) $GB^{id}_{0,p,q}(g') \in L^\infty(\mathbb{D}, dA)$ *if and only if*

$$\psi_\omega(z)|g'(z)| \in L^\infty(\mathbb{D}, dA).$$

  (b) $\lim_{|z|\to 1} GB^{id}_{0,p,q}(g') = 0$ *if and only if*

$$\lim_{|z|\to 1} \psi_\omega(z)|g'(z)| = 0.$$

(II) *Let* $\omega \in \mathcal{W}$ *with*

$$\Delta\phi(z) \asymp ((1-|z|)^t \psi_\omega(z))^{-1}, \; z \in \mathbb{D}, \textit{ for some } t \geq 1. \tag{1.12}$$

  For $p < q$, *the following statements are equivalent:*
  (c) $GB^{id}_{0,p,q}(g') \in L^\infty(\mathbb{D}, dA)$.
  (d) *The function* g *is constant.*

(III) *For* $q < p$,

$$GB^{id}_{0,p,q}(g') \in L^{p/(p-q)}(\mathbb{D}, d\lambda) \implies g \in A^{pq/(p-q)}(\omega).$$

1.2. Outline

In Section 2 we provide the basic definitions and results that are needed to deal with the weights ω in $\mathcal{W}$, and consider useful estimates for the reproducing kernel of A^p_ω. In Section 3, we recall known geometric characterizations of Carleson measures, and in Section 4 we establish embedding theorems of S^p_ω into $L^q(\mathbb{D}, d\mu)$, for $0 < p, q \leq \infty$ and $\omega \in \mathcal{W}$, where

$$S^p_\omega := \left\{ f \in H(\mathbb{D}) : \|f\|_{S^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \frac{\omega(z)^{p/2}}{(1+\varphi'(z))^p} \, dA(z) < \infty \right\} \tag{1.13}$$

and

$$S^\infty_\omega := \left\{ f \in H(\mathbb{D}) : \|f\|_{S^\infty_\omega} = \sup_{z \in \mathbb{D}} |f(z)| \frac{\omega(z)^{1/2}}{1+\varphi'(z)} < \infty \right\}. \tag{1.14}$$

In Section 5, we prove Theorems 1.1 and 1.2 using the embedding theorems, the strong decay of the weights $e^{-2\varphi}$ and the following Littlewood-Paley type formulas (see (9.3) of [3] and [13]):

$$\|f\|^p_{A^p_\omega} \asymp |f(0)| + \int_{\mathbb{D}} |f'(z)|^p \frac{\omega(z)^{p/2}}{(1+\varphi'(z))^p} \, dA(z), \tag{1.15}$$

$$\|f\|_{A^\infty_\omega} \asymp |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| \frac{\omega(z)^{1/2}}{(1+\varphi'(z))}. \tag{1.16}$$

Finally, Proposition 1.3 and Corollaries 1.5 and 1.6 are proved in Section 6.

Throughout the paper, we use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \leq Cb$. By $a \asymp b$ we mean that $a \lesssim b$ and $b \lesssim a$. For simplicity, we write L^p_ω and A^p_ω for $L^p(\mathbb{D}, \omega^{p/2} \, dA)$ and $A^p(\mathbb{D}, \omega^{p/2} \, dA)$, respectively.

2. Preliminaries and basic properties

A positive function τ on $\mathbb{D}$ is said to be of class $\mathcal{L}$ if there are two constants c_1 and c_2 such that

$$\tau(z) \leq c_1 (1 - |z|) \quad \text{for all } z \in \mathbb{D} \tag{2.1}$$

and

$$|\tau(z) - \tau(\zeta)| \leq c_2 |z - \zeta| \quad \text{for all } z, \zeta \in \mathbb{D}. \tag{2.2}$$

For such c_1 and c_2, we set

$$m_\tau := \tfrac{1}{4}\min(1, c_1^{-1}, c_2^{-1}).$$

Given $a \in \mathbb{D}$ and $\delta > 0$, we denote by $D_\delta(a)$ the Euclidean disc centered at a with radius $\delta\tau(a)$. It follows from (2.1) and (2.2) (see [15, Lemma 2.1]) that if $\tau \in \mathcal{L}$ and $z \in D_\delta(a)$, then

$$\frac{1}{2}\,\tau(a) \le \tau(z) \le 2\,\tau(a), \tag{2.3}$$

whenever $\delta \in (0, m_\tau)$. These inequalities will be used frequently in what follows.

Definition 2.1. We say that a weight ω is of *class* $\mathcal{L}^*$ if it is of the form $\omega = e^{-2\varphi}$, where $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$, and $\big(\Delta\varphi(z)\big)^{-1/2} \asymp \tau(z)$, with τ being a function in the class $\mathcal{L}$. Here Δ denotes the classical Laplace operator.

It is straightforward to see that $\mathcal{W} \subset \mathcal{L}^*$. The following result (see [15, Lemma 2.2]) implies that the point evaluation functional at each $z \in \mathbb{D}$ is bounded on A^2_ω.

Lemma A. *Let $\omega \in \mathcal{L}^*$, $0 < p < \infty$, and $z \in \mathbb{D}$. If $\beta \in \mathbb{R}$, there exists $M \ge 1$ such that*

$$|f(z)|^p\omega(z)^\beta \le \frac{M}{\delta^2\tau(z)^2}\int\limits_{D_\delta(z)} |f(\xi)|^p\omega(\xi)^\beta \, dA(\xi)$$

for all $f \in H(\mathbb{D})$ and all sufficiently small $\delta > 0$.

Using the preceding lemma and the fact that there exists $r_0 \in [0,1)$ such that for all $a \in \mathbb{D}$ with $1 > |a| > r_0$, and any $\delta > 0$ small enough we have

$$\varphi'(a) \asymp \varphi'(z), \quad z \in D_\delta(a)$$

(see statement (d) in [3, Lemma 32]), one has

$$|f(z)|^p\frac{\omega(z)^\beta}{(1+\varphi'(z))^\gamma} \lesssim \frac{1}{\delta^2\tau(z)^2}\int\limits_{D_\delta(z)} |f(\xi)|^p\frac{\omega(\xi)^\beta}{(1+\varphi'(\xi))^\gamma}dA(\xi), \tag{2.4}$$

for $\beta, \gamma \in \mathbb{R}$.

The next lemma provides upper estimates for the derivatives of functions in A^p_ω. Its proof is similar to the case of doubling measures $\Delta\varphi$ in Lemma 19 of [10], and it can be found in the following form in [5,14].

Lemma B. *Let $\omega \in \mathcal{L}^*$ and $0 < p < \infty$. For any $\delta_0 > 0$ sufficiently small there exists a constant $C(\delta_0) > 0$ such that*

$$|f'(z)|^p\omega(z)^{p/2} \le \frac{C(\delta_0)}{\tau(z)^{2+p}}\left(\int\limits_{D(\delta_0\tau(z)/2)} |f(\xi)|^p\,\omega(\xi)^{p/2}dA(\xi)\right)^{1/p},$$

for all $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

The following lemma on coverings is due to Oleinik [14].

Lemma C. *Let τ be a positive function on $\mathbb{D}$ of class $\mathcal{L}$, and let $\delta \in (0, m_\tau)$. Then there exists a sequence of points $\{z_n\} \subset \mathbb{D}$ such that the following conditions are satisfied:*

(*i*) $z_n \notin D_\delta(z_k)$, $n \neq k$.
(*ii*) $\bigcup_n D_\delta(z_n) = \mathbb{D}$.
(*iii*) $\tilde{D}_\delta(z_n) \subset D_{3\delta}(z_n)$, *where* $\tilde{D}_\delta(z_n) = \bigcup_{z\in D_\delta(z_n)} D_\delta(z)$, $n \in \mathbb{N}$.
(*iv*) $\{D_{3\delta}(z_n)\}$ *is a covering of $\mathbb{D}$ of finite multiplicity N.*

The multiplicity N in the previous lemma is independent of δ, and it is easy to see that one can take, for example, $N = 256$. Any sequence satisfying the conditions in Lemma C will be called a (δ, τ)-*lattice*. Note that $|z_n| \to 1^-$ as $n \to \infty$. In what follows, the sequence $\{z_n\}$ will always refer to the sequence chosen in Lemma C.

2.1. Reproducing kernel estimates

The following norm estimates for the reproducing kernel K_z valid for all $z \in \mathbb{D}$ can be found in [2,7,15] when $p = 2$ and in [5] when $p > 0$, while for the estimate for the points close to the diagonal, see [8, Lemma 3.6].

Theorem A. *Let K_z be the reproducing kernel of A^2_ω. Then*

(a) *For $\omega \in \mathcal{W}$ and $0 < p < \infty$, one has*

$$\|K_z\|_{A^p_\omega} \asymp \omega(z)^{-1/2}\,\tau(z)^{2(1-p)/p}, \qquad z \in \mathbb{D}. \tag{2.5}$$

$$\|K_z\|_{A^\infty_\omega} \asymp \omega(z)^{-1/2}\,\tau(z)^{-2}, \qquad z \in \mathbb{D}. \tag{2.6}$$

(b) *For all sufficiently small $\delta \in (0, m_\tau)$ and $\omega \in \mathcal{W}$, one has*

$$|K_z(\zeta)| \asymp \|K_z\|_{A^2_\omega} \cdot \|K_\zeta\|_{A^2_\omega}, \qquad \zeta \in D_\delta(z). \tag{2.7}$$

The next lemma generalizes the statement (a) of the above theorem. For the proof, see [1].

Lemma D. *Let K_z be the reproducing kernel of A^2_ω where ω is a weight in the class $\mathcal{W}$. For each $z \in \mathbb{D}$, $0 < p < \infty$ and $\beta \in \mathbb{R}$, one has*

$$\int_{\mathbb{D}} |K_z(\xi)|^p \, \omega(\xi)^{p/2} \, \tau(\xi)^{\beta} \, dA(\xi) \asymp \omega(z)^{-p/2} \, \tau(z)^{2(1-p)+\beta}. \tag{2.8}$$

The following result gives estimates for the normalized reproducing kernel $k_{p,z}$ in A^p_ω defined by

$$k_{p,z} = K_z / \|K_z\|_{A^p_\omega}$$

for $z \in \mathbb{D}$.

Lemma 2.2. *Let $\omega \in \mathcal{W}$. Then*

(a) *For each $z \in \mathbb{D}$, $0 < p \le \infty$, and $0 < q < \infty$,*

$$|k_{p,z}(\zeta)|^q \asymp \tau(z)^{2(1-\frac{q}{p})} |k_{q,z}(\zeta)|^q, \qquad \zeta \in \mathbb{D}. \tag{2.9}$$

(b) *For $q = \infty$,*

$$|k_{p,z}(\zeta)| \asymp \tau(z)^{-2/p} |k_{q,z}(\zeta)|, \qquad \zeta \in \mathbb{D}.$$

(c) *For all $\delta \in (0, m_\tau)$ sufficiently small,*

$$|k_{p,z}(\zeta)|^p \, \omega(\zeta)^{p/2} \asymp \tau(z)^{-2}, \qquad \zeta \in D_\delta(z). \tag{2.10}$$

Proof. The proof is immediate from Theorem A. □

2.2. Test functions and some estimates

The following result on test functions was obtained in [15] and Lemma 3.3 in [2]. Without loss of generality, we modified the original version by taking $\omega(z)^{p/2}$ instead of $\omega(z)$ when $0 < p < \infty$.

Lemma E. *Let $n \in \mathbb{N} \setminus \{0\}$ and $\omega \in \mathcal{W}$. There is a number $\rho_0 \in (0, 1)$ such that for each $a \in \mathbb{D}$ with $|a| > \rho_0$ there is a function $F_{a,n}$ analytic in $\mathbb{D}$ with*

$$|F_{a,n}(z)| \, \omega(z)^{1/2} \asymp 1 \quad \text{if} \quad |z - a| < \tau(a), \tag{2.11}$$

and

$$|F_{a,n}(z)| \, \omega(z)^{1/2} \lesssim \min\left(1, \frac{\min\left(\tau(a), \tau(z)\right)}{|z - a|}\right)^{3n}, \quad z \in \mathbb{D}. \tag{2.12}$$

Moreover,

(a) *For* $0 < p < \infty$, *the function* $F_{a,n}$ *belongs to* $A^p(\omega)$ *with*

$$\|F_{a,n}\|_{A^p_\omega} \asymp \tau(a)^{2/p}.$$

(b) *For* $p = \infty$, *the function* $F_{a,n}$ *belongs to* A^∞_ω *with*

$$\|F_{a,n}\|_{A^\infty_\omega} \asymp 1.$$

As a consequence, we have the following pointwise estimates for the derivative of the test functions $F_{a,n}$. Its proof is a simple application of (2.11).

Lemma 2.3. *Let* $n \in \mathbb{N} \setminus \{0\}$ *and* $\omega \in \mathcal{W}$. *For any* $\delta > 0$ *small enough,*

$$|F'_{a,n}(z)|\,\omega(z)^{1/2} \asymp 1 + \varphi'(z), \quad z \in D_\delta(a). \tag{2.13}$$

The next Proposition is a partial result about the atomic decomposition on A^p_ω and its proof follows easily from Lemma E.

Proposition 2.4. *Let* $n \geq 2$ *and* $\omega \in \mathcal{W}$. *Let* $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$ *be the sequence defined in Lemma C.*

(a) *For* $0 < p < \infty$, *the function given by*

$$F(z) := \sum_{k=0}^{\infty} \lambda_k \, \frac{F_{z_k,n}(z)}{\tau(z_k)^{2/p}}$$

belongs to A^p_ω *for every sequence* $\lambda = \{\lambda_k\} \in \ell^p$. *Moreover,*

$$\|F\|_{A^p_\omega} \lesssim \|\lambda\|_{\ell^p}.$$

(b) *For* $p = \infty$, *the function given by*

$$F(z) := \sum_{k=0}^{\infty} \lambda_k \, F_{z_k,n}(z)$$

belongs to A^∞_ω *for every sequence* $\lambda = \{\lambda_k\} \in \ell^\infty$. *Moreover,*

$$\|F\|_{A^\infty_\omega} \lesssim \|\lambda\|_{\ell^\infty}.$$

Proof. The proof of (a) can be found in [15, Proposition 2]. To prove (b), estimate the norm of F as follows

$$\begin{aligned}\|F\|_{A^\infty_\omega} &= \sup_{z\in\mathbb{D}} |F(z)|\omega(z)^{1/2} \\ &\lesssim \|\lambda\|_{\ell^\infty} \sum_{k=0}^{\infty} |F_{z_k,n}(z)|\omega(z)^{1/2} \\ &= \|\lambda\|_{\ell^\infty} \left(\sum_{z_k\in D_\delta(z)} |F_{z_k,n}(z)|\omega(z)^{1/2} + \sum_{z_k\notin D_\delta(z)} |F_{z_k,n}(z)|\omega(z)^{1/2} \right)\end{aligned}$$

Now, using (2.11) and (iv) of Lemma C, we have

$$\sum_{z_k\in D_\delta(z)} |F_{z_k,n}(z)|\omega(z)^{1/2} \lesssim 1. \tag{2.14}$$

It remains to show that

$$\sum_{z_k\notin D_\delta(z)} |F_{z_k,n}(z)|\omega(z)^{1/2} \lesssim 1.$$

Indeed, by Hölder's inequality, we have

$$\sum_{z_k\notin D_\delta(z)} |F_{z_k,n}(z)|\omega(z)^{1/2} \le I(z)\cdot II(z), \tag{2.15}$$

where

$$I(z) = \sum_{z_k\notin D_\delta(z)} \min\big(\tau(z_k),\tau(z)\big)^2 |F_{z_k,n}(z)|\omega(z)^{1/2},$$

and

$$II(z) = \sum_{z_k\notin D_\delta(z)} \frac{|F_{z_k,n}(z)|\omega(z)^{1/2}}{\min\big(\tau(z_k),\tau(z)\big)^2}.$$

First we look for the upper bound of $I(z)$. To do this, we need to consider the covering of $\{\xi\in\mathbb{D} : |z-\xi| > \delta\tau(z)\}$ given by

$$R_j(z) = \{\xi\in\mathbb{D} : 2^j\delta\tau(z) < |z-\xi| \le 2^{j+1}\delta\tau(z)\}, \qquad j=0,1,2,\dots$$

and observe that, using (A) of properties of τ, it is easy to see that, for $j=0,1,2,\dots$,

$$D_\delta(z_k)\subset D_r(z), \quad \text{if} \;\; z_k\in D_t(z) \text{ with } r=5\delta 2^j \text{ and } t=\delta 2^{j+1}.$$

This fact together with the finite multiplicity of the covering (see Lemma C) gives

$$\sum_{z_k \in R_j(z)} \tau(z_k)^2 \lesssim A(D_r(z)) \lesssim 2^{2j}\tau(z)^2.$$

Therefore, by (2.12), we have

$$\begin{aligned} I(z) &\le \sum_{z_k \notin D_\delta(z)} \tau(z_k)^2 |F_{z_k,n}(z)| \omega(z)^{1/2} \\ &\lesssim \sum_{z_k \notin D_\delta(z)} \tau(z_k)^2 \min\left(1, \frac{\min\left(\tau(z_k), \tau(z)\right)}{|z - z_k|}\right)^{3n} \\ &\le \tau(z)^{3n} \sum_{j=0}^{\infty} \sum_{z_k \in R_j(z)} \frac{\tau(z_k)^2}{|z - z_k|^{3n}} \\ &\lesssim \sum_{j=0}^{\infty} 2^{-3nj} \sum_{z_k \in R_j(z)} \tau(z_k)^2 \\ &\lesssim \tau(z)^2 \sum_{j=0}^{\infty} 2^{(2-3n)j} \lesssim \tau(z)^2. \end{aligned} \tag{2.16}$$

To obtain an upper estimate for (II), notice that since $n \ge 2$, (2.12) implies that

$$\begin{aligned} II(z) &\lesssim \sum_{z_k \notin D_\delta(z)} \frac{\min\left(\tau(z_k), \tau(z)\right)^{3n-2}}{|z - z_k|^{3n}} \\ &\le \tau(z)^{3n-4} \sum_{j=0}^{\infty} \sum_{z_k \in R_j(z)} \frac{\tau(z_k)^2}{|z - z_k|^{3n}} \\ &\lesssim \tau(z)^{-4} \sum_{j=0}^{\infty} 2^{-3nj} \sum_{z_k \in R_j(z)} \tau(z_k)^2 \\ &\lesssim \tau(z)^{-2} \sum_{j=0}^{\infty} 2^{(2-3n)j} \lesssim \tau(z)^{-2}. \end{aligned}$$

Combining this and (2.16) with (2.15) completes the proof. □

3. Geometric characterizations of Carleson measures

Let μ be a positive measure on $\mathbb{D}$. Denote by $\widehat{\mu_\delta}$ the averaging function defined as

$$\widehat{\mu_\delta}(z) = \mu(D_\delta(z)) \cdot \tau(z)^{-2}, \quad z \in \mathbb{D},$$

and define the general Berezin transform of μ by

$$G_t(\mu)(z) = \int_{\mathbb{D}} |k_{t,z}(\zeta)|^t \, \omega(\zeta)^{t/2} \, d\mu(\zeta),$$

for every $t > 0$ and $z \in \mathbb{D}$.

In this section we recall recent characterizations of q-Carleson measures for A^p_ω for any $0 < p, q \leq \infty$ in terms of the averaging function $\widehat{\mu_\delta}$ and the general Berezin transform $G_t(\mu)$. For the proofs of all theorems in this section, see Section 3 of [1].

3.1. Carleson measures

We begin with the definition of q-Carleson measures.

Definition 3.1. Let μ be a positive measure on $\mathbb{D}$ and fix $0 < p, q < \infty$. We say that μ is a q-Carleson measure for A^p_ω if the inclusion $I_\mu : A^p_\omega \longrightarrow L^q_\omega$ is bounded.

The following theorem characterizes the q-Carleson measures when $0 < p \leq q < \infty$.

Theorem B. *Let μ be a finite positive Borel measure on $\mathbb{D}$. Assume $0 < p \leq q < \infty$, $s = p/q$, $0 < t < \infty$. The following conditions are equivalent:*

(a) *The measure μ is a q-Carleson measure for A^p_ω.*
(b) *The function*

$$\tau(z)^{2(1-1/s)} G_t(\mu)(z)$$

belongs to $L^\infty(\mathbb{D}, dA)$.

(c) *The function*

$$\tau(z)^{2(1-1/s)} \widehat{\mu_\delta}(z)$$

belongs to $L^\infty(\mathbb{D}, dA)$ for any sufficiently small $\delta > 0$.

Now we characterize q-Carleson measures when $0 < q < p < \infty$.

Theorem C. *Let μ be a finite positive Borel measure on $\mathbb{D}$. Assume $0 < q < p < \infty$ and $s = p/q$. The following conditions are all equivalent:*

(a) *The measure μ is a q-Carleson measure for A^p_ω.*
(b) *For any (or some) $r > 0$, we have*

$$\widehat{\mu_r} \in L^{p/(p-q)}(\mathbb{D}, dA).$$

(c) *For any $t > 0$,*

$$G_t(\mu) \in L^{p/(p-q)}(\mathbb{D}, dA).$$

3.2. Vanishing Carleson measures

Definition 3.2. Let μ be a positive measure on $\mathbb{D}$ and fix $0 < p, q < \infty$. We say that μ is a vanishing q-Carleson measure for A^p_ω if the inclusion $I_\mu : A^p_\omega \longrightarrow L^q_\omega$ is compact, or equivalently, if

$$\int_{\mathbb{D}} |f_n(z)|^q \, \omega(z)^{q/2} \, d\mu(z) \to 0$$

whenever f_n is bounded in A^p_ω and converges to zero uniformly on each compact subsets of $\mathbb{D}$.

The following three theorems characterize vanishing q-Carleson measures for A^p_ω when $0 < p \leq \infty$ and $0 < q < \infty$.

Theorem D. *Given $\tau \in \mathcal{L}^*$, let μ be a finite positive Borel measure on $\mathbb{D}$. Assume $0 < p \leq q < \infty$, $s = p/q$, $0 < t < \infty$. The following statements are all equivalent:*

(a) *μ is a vanishing q-Carleson measure for A^p_ω.*
(b) *$\tau(z)^{2(1-1/s)} G_t(\mu)(z) \to 0$ as $|z| \to 1^-$.*
(c) *$\tau(z)^{2(1-1/s)} \widehat{\mu_\delta}(z) \to 0$ as $|z| \to 1^-$, for any small enough $\delta > 0$.*

Theorem E. *Given $\tau \in \mathcal{L}^*$, let μ be a finite positive Borel measure on $\mathbb{D}$. Assume $0 < q < \infty$. The following conditions are all equivalent:*

(a) *μ is a q-Carleson measure for A^∞_ω.*
(b) *μ is a vanishing q-Carleson measure for A^∞_ω.*
(c) *For any sufficiently small $\delta > 0$, we have*

$$\widehat{\mu_\delta} \in L^1(\mathbb{D}, dA).$$

(d) *For any $t > 0$, we have*

$$G_t(\mu) \in L^1(\mathbb{D}, dA).$$

Theorem 3.3. *Given $\tau \in \mathcal{L}^*$, let μ be a finite positive Borel measure on $\mathbb{D}$. Assume that $0 < q < p < \infty$. The following statements are equivalent:*

(a) *μ is a q-Carleson measure for A^p_ω.*
(b) *μ is a vanishing q-Carleson measure for A^p_ω.*

4. Embedding theorems

In this section we establish embedding theorems of S^p_ω into $L^q(\mathbb{D}, d\mu)$ for $0 < p, q \leq \infty$ and $\omega \in \mathcal{W}$, where S^p_ω are given in (1.13) and (1.14). We start with the case $0 < p \leq q < \infty$.

Lemma 4.1. *Let $\omega \in \mathcal{W}$ and $0 < p \leq q < \infty$. Let μ be a positive Borel measure on $\mathbb{D}$. Then*

(a) *$I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is bounded if and only if for each $\delta >$ small enough,*

$$K_{\mu,\omega}(z) = \sup_{z\in\mathbb{D}} \frac{1}{\tau(z)^{2q/p}} \int\limits_{D_\delta(z)} (1 + \varphi'(\xi))^q \omega(\xi)^{-q/2}\, d\mu(\xi) < \infty. \tag{4.1}$$

(b) *$I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is compact if and only if*

$$\lim_{|z|\to 1^-} \frac{1}{\tau(z)^{2q/p}} \int\limits_{D_\delta(z)} (1 + \varphi'(\xi))^q \omega(\xi)^{-q/2}\, d\mu(\xi) = 0. \tag{4.2}$$

Proof. Suppose first that the condition (4.1) holds. Then, by Lemma C and (2.4), we get

$$\|f\|^q_{L^q(\mathbb{D},d\mu)} = \int\limits_{\mathbb{D}} |f(z)|^q\, d\mu(z) \leq \sum_{k=0}^{\infty} \int\limits_{D_\delta(z_k)} |f(z)|^q\, d\mu(z)$$

$$= \sum_{k=0}^{\infty} \int\limits_{D_\delta(z_k)} |f(z)|^q \frac{\omega(z)^{q/2}}{(1+\varphi'(z))^q} (1+\varphi'(z))^q\, \omega(z)^{-q/2}\, d\mu(z)$$

$$\lesssim \sum_{k=0}^{\infty} \int\limits_{D_\delta(z_k)} \left(\frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1+\varphi'(s))^p} dA(s) \right)^{q/p} (1+\varphi'(z))^q \omega(z)^{-q/2} d\mu(z)$$

$$\lesssim \sum_{k=0}^{\infty} \left(\int\limits_{D_{3\delta}(z_k)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1+\varphi'(s))^p}\, dA(s) \right)^{q/p} \int\limits_{D_\delta(z_k)} \frac{(1+\varphi'(z))^q \omega(z)^{-q/2}}{\tau(z)^{2q/p}} d\mu(z),$$

for small enough $\delta > 0$. By applying our assumption, we have

$$\int\limits_{\mathbb{D}} |f(z)|^q\, d\mu(z) \lesssim K_{\mu,\omega} \sum_{k=0}^{\infty} \left(\int\limits_{D_{3\delta}(z_k)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1+\varphi'(s))^p}\, dA(s) \right)^{q/p}.$$

Using a similar argument as in the proof of Theorem 1.1 in [14], Minkowski's inequality and the finite multiplicity N of the covering $\{D_{3\delta}(z_k)\}$, we get

$$\|f\|^q_{L^q(\mathbb{D},d\mu)} \lesssim K_{\mu,\omega} \left(\sum_{k=0}^{\infty} \int\limits_{D_{3\delta}(z_k)} |f(s)|^p \frac{\omega(s)^{p/2}}{(1+\varphi'(s))^p}\, dA(s) \right)^{q/p} \lesssim K_{\mu,\omega}\, N^{q/p}\, \|f\|^q_{S^p_\omega}.$$

This proves that the embedding $I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is bounded with $\|I_\mu\|^q_{L^q(\mathbb{D},d\mu)} \leq K_{\mu,\omega}$.

Conversely, suppose that $I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is bounded. Let $a \in \mathbb{D}$ with $|a| \geq \rho_0$ that is defined in Lemma E. By Lemma 2.3,

$$|F'_{a,n}(z)|\, \omega(z)^{1/2} \asymp (1 + \varphi'(z)),\ \ z \in D_\delta(a),$$

(where $F_{a,n}$ is the test function in Lemma E), and so

$$\int\limits_{D_\delta(a)} (1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}\, d\mu(z) \lesssim \int\limits_{D_\delta(a)} |F'_{a,n}(z)|^q\, d\mu(z) \lesssim \int\limits_{\mathbb{D}} |F'_{a,n}(z)|^q\, d\mu(z).$$

Using our assumption, (a) of Lemma E, and (1.15), we obtain

$$\begin{aligned}\int\limits_{D_\delta(a)} (1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}\, d\mu(z) &\lesssim \|I_\mu\|^q\, \|F'_{a,n}\|^q_{S^p_\omega} \\ &\leq \|I_\mu\|^q\, \|F_{a,n}\|^q_{A^p_\omega} \asymp \|I_\mu\|^q\, \tau(a)^{2q/p}.\end{aligned}$$

Then dividing both sides by $\tau(a)^{2q/p}$ gives

$$\frac{1}{\tau(a)^{2q/p}} \int\limits_{D_\delta(a)} (1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}\, d\mu(z) \leq \|I_\mu\|^q < \infty,$$

and so

$$\sup_{a\in\mathbb{D}} \frac{1}{\tau(a)^{2q/p}} \int\limits_{D_\delta(a)} (1+\varphi'(z))^q\, \omega(z)^{-q/2}\, d\mu(z) \leq \|I_\mu\|^q < \infty,$$

which means that $K_{\mu,\omega} \lesssim \|I_\mu\|^q$.

To prove (b), suppose that $I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is compact. Consider the function

$$f_{a,n}(z) := \frac{F_{a,n}(z)}{\tau(a)^{2q/p}}, \quad \text{for } |a| \geq \rho_0.$$

As in the proof of Theorem 1 of [15] and using Lemma E, we can show that the function $f_{a,n}$ is bounded and converges to zero uniformly on compact subsets of $\mathbb{D}$ when $|a| \to 1^-$.

Therefore, by Lemma B, $f'_{a,n}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$.

$$\frac{1}{\tau(a)^{2q/p}} \int\limits_{D_\delta(a)} (1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}\, d\mu(z) \lesssim \int\limits_{D_\delta(a)} |f'_{a,n}(z)|^q \, d\mu(z)$$
$$\leq \int\limits_{\mathbb{D}} |f'_{a,n}(z)|^q \, d\mu(z) = \|I_\mu f'_{a,n}\|_{L^q(\mu)}.$$

Since I_μ is compact,

$$\lim_{|a|\to 1^-} \|f'_{a,n}\|_{L^q(\mu)} = 0,$$

and so

$$\lim_{|a|\to 1^-} \frac{1}{\tau(a)^{2q/p}} \int\limits_{D_\delta(a)} (1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}\, d\mu(z) = 0.$$

This shows that (4.2) holds.

Conversely, suppose that (4.2) holds. Let $\{f_n\} \subset S^p_\omega$ be a bounded sequence converging to zero uniformly on compact subsets of $\mathbb{D}$ and $\{z_k\}$ be a (δ, τ)-lattice. To prove that I_μ is compact, it suffices to show that $\|f_n\|_{L^q(\mu)} \to 0$. By the assumption, given any $\varepsilon > 0$, there exists $0 < r_1 < 1$ with

$$\frac{1}{\tau(a)^{2q/p}} \int\limits_{D_\delta(a)} (1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}\, d\mu(z) < \varepsilon, \quad r_1 < |a| < 1. \tag{4.3}$$

Observe that there is $r_1 < r_2 < 1$ such that if a point z_j of the sequence $\{z_k\}$ belongs to $\{z \in \mathbb{D} : |z| \leq r_1\}$, then $D_\delta(z_j) \subset \{z \in \mathbb{D} : |z| \leq r_2\}$. Therefore, since $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, there exists an integer n_0 such that

$$|f_n(z)| < \varepsilon, \quad \text{for } |z| \leq r_2 \text{ and } n \geq n_0.$$

We split the integration of this function into two parts: the first integration is over $|z| \leq r_2$ and the other integration is over $|z| \geq r_2$. On the one hand,

$$\int\limits_{|z|\leq r_2} |f_n(z)|^q \, d\mu(z) < \varepsilon^q. \tag{4.4}$$

On the other hand, by Lemma C and Lemma B, we obtain

$$\begin{aligned}\int\limits_{|z|>r_2} |f_n(z)|^q\,d\mu(z) &\le \sum_{|z_k|>r_1} \int\limits_{D_\delta(z_k)} |f_n(z)|^q\,d\mu(z)\\
&\lesssim \sum_{|z_k|>r_1} \int\limits_{D_\delta(z_k)} \left(\frac{1}{\tau(z_k)^2} \int\limits_{D_\delta(z)} |f_n(s)|^p \frac{\omega(s)^{p/2}}{(1+\varphi'(s))^p}\,dA(s) \right)^{q/p} (1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}\,d\mu(z)\\
&\lesssim \sum_{k=0}^{\infty} \left(\int\limits_{D_{3\delta}(z_k)} |f_n(s)|^p \frac{\omega(s)^{p/2}}{(1+\varphi'(s))^p}\,dA(s) \right)^{q/p} \int\limits_{D_\delta(z_k)} \frac{(1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}}{\tau(z_k)^{2q/p}}\,d\mu(z)\\
&\lesssim \varepsilon \|f_n\|^q_{S^p_\omega} \sup_{|z_k|>r_1} \frac{1}{\tau(z_k)^{2q/p}} \int\limits_{D_\delta(z_k)} (1+\varphi'(z))^q \omega(z)^{-\frac{q}{2}}\,d\mu(z) \lesssim \varepsilon\, \|f_n\|^q_{S^p_\omega} \lesssim \varepsilon.\end{aligned}$$

These together with (4.4) show that $I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is compact. □

To characterize boundedness and compactness of $I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ with $0 < q < p < \infty$, consider the function $F_{\delta,\mu}(\varphi)$ defined by

$$F_{\delta,\mu}(\varphi)(z) := \frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} (1+\varphi'(\xi))^q \omega(\xi)^{-q/2}\,d\mu(\xi). \tag{4.5}$$

We use Luecking's approach in [9] based on Khinchine's inequality. Recall that Rademacher functions R_n are defined by

$$R_0(t) = \begin{cases} 1 & \text{if } 1 \le t - [t] < 1/2\\ -1 & \text{if } 1/2 \le t - [t] < 1; \end{cases}$$
$$R_n(t) = R_0(2^n t), \quad n \ge 1,$$

where $[t]$ denotes the largest integer not exceeding t.

Lemma F *(Khinchine's inequality [9]). For $0 < p < \infty$, there exists a positive constant C_p such that*

$$C_p^{-1}\Big(\sum_{k=1}^n |\lambda_k|^2\Big)^{p/2} \le \int\limits_0^1 \Big|\sum_{k=1}^n \lambda_k R_k(t)\Big|^p dt \le C_p\Big(\sum_{k=1}^n |\lambda_k|^2\Big)^{p/2},$$

for all $n \in \mathbb{N}$ and $\{\lambda_k\}_{k=1}^n \subset \mathbb{C}$.

Lemma 4.2. *Let $\omega \in \mathcal{W}$ and $0 < q < p < \infty$. Let μ be a finite positive Borel measure on $\mathbb{D}$. Then, the following statements are equivalent:*

(a) *The operator $I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is bounded.*
(b) *The operator $I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is compact.*

(c) *The function*

$$F_{\delta,\mu}(\varphi) \in L^{p/(p-q)}(\mathbb{D}, dA). \tag{4.6}$$

Proof. The implication (b) $\Rightarrow$ (a) is obvious. To prove that (a) implies (c), suppose that the operator $I_\mu : S^p_\omega \to L^q(\mathbb{D}, d\mu)$ is bounded. Let $\{z_k\}$ be a (δ,τ)-lattice on $\mathbb{D}$. Corresponding to each $\lambda = \{\lambda_m\}_m \in \ell^p$, we consider

$$f(z) = \sum_{|z_m|\geq\rho_0} \lambda_m f_{z_m,n}(z),$$

where $f_{z_m,n}(z) = \frac{F_{z_m,n}(z)}{\tau(z_m)^{2/p}}$ and $0 < \rho_0 < 1$ as in Lemma E. By Proposition 2.4 and (1.15),

$$\|f'\|_{S^p_\omega} \lesssim \|f\|_{A^p_\omega} \lesssim \|\lambda\|_{\ell^p}.$$

Note that as an application of Khinchine's inequality (Lemma F), replace λ_m with the Rademacher functions $R_m(t)\lambda_m$, and then integrate with respect to t from 0 and 1, which yields

$$\left(\sum_{|z_m|\geq\rho_0} \left|\lambda_m\right|^2 \left|f'_{z_m,n}(z)\right|^2\right)^{q/2} \lesssim \int_0^1 \left|\sum_{|z_m|\geq\rho_0} R_m(t)\lambda_m\, f'_{z_m,n}(z)\right|^q dt$$

and so

$$\begin{aligned}
&\int_{\mathbb{D}} \left(\sum_{|z_m|\geq\rho_0} |\lambda_m|^2 |f'_{z_m,n}(z)|^2\,\omega(z)\right)^{q/2} d\mu(z) \\
&\lesssim \int_{\mathbb{D}} \int_0^1 \left|\sum_{z_m:|z_m|\geq\rho_0} R_m(t)\lambda_m\, f'_{z_m,n}(z)\right|^q \omega(z)^{q/2} dt\, d\mu(z) \\
&= \int_0^1 \int_{\mathbb{D}} \left|\sum_{z_m:|z_m|\geq\rho_0} R_m(t)\lambda_m\, f'_{z_m,n}(z)\right|^q \omega(z)^{q/2} d\mu(z)\, dt \\
&\lesssim \int_0^1 \|f'\|^q_{S^p_\omega}\, dt = \|f'\|^q_{S^p_\omega} \lesssim \|f\|^q_{A^p_\omega} \lesssim \|\lambda\|^q_{\ell^p}.
\end{aligned} \tag{4.7}$$

By Lemmas C and 2.3,

$$\sum_{|z_m|\geq\rho_0} \frac{|\lambda_m|^q}{\tau(z_m)^{2/p}} \int_{D_{3\delta}(z_m)} (1+\varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}}\, d\mu(\xi)$$

$$\lesssim \sum_{|z_m|\geq\rho_0} |\lambda_m|^q \int_{D_{3\delta}(z_m)} |f'_{z_m,n}(\xi)|^q \, d\mu(\xi)$$

$$= \int_{\mathbb{D}} \sum_{|z_m|\geq\rho_0} |\lambda_m|^q \, |f'_{z_m,n}(\xi)|^q \chi_{D_{3\delta}(z_m)}(\xi) \, d\mu(\xi),$$

where $\chi_{D_{3\delta}(z_m)}(\xi)$ denotes the characteristic function of the set $D_{3\delta}(z_m)$. Now, by the fact that $\sum_s^\infty z_m^k \leq \left(\sum_s^\infty z_m\right)^k$, $k \geq 1, z_m \geq 0$ for $q \geq 2$, we get

$$\int_{\mathbb{D}} \sum_{|z_m|\geq\rho_0} |\lambda_m|^q \, |f'_{z_m,n}(\xi)|^q \chi_{D_{3\delta}(z_m)}(\xi) \, d\mu(\xi)$$

$$= \int_{\mathbb{D}} \sum_{|z_m|\geq\rho_0} \left(|\lambda_m|^2 \, |f'_{z_m,n}(\xi)|^2 \chi_{D_{3\delta}(z_m)}(\xi)\right)^{q/2} d\mu(\xi)$$

$$\lesssim \int_{\mathbb{D}} \Big(\sum_{|z_m|\geq\rho_0} |\lambda_m|^2 \, |f'_{z_m,n}(\xi)|^2 \Big)^{q/2} d\mu(\xi).$$

For $q < 2$, by Hölder's inequality and Lemma C, we get

$$\int_{\mathbb{D}} \sum_{|z_m|\geq\rho_0} |\lambda_m|^q \, |f'_{z_m,n}(\xi)|^q \chi_{D_{3\delta}(z_m)}(\xi) \, d\mu(\xi)$$

$$\leq \int_{\mathbb{D}} \left(\sum_{|z_m|\geq\rho_0} |\lambda_m|^2 \, |f'_{z_m,n}(\xi)|^2 \right)^{q/2} \left(\sum_{|z_m|\geq\rho_0} \chi_{D_{3\delta}(z_m)}(\xi) \right)^{1-\frac{q}{2}} d\mu(z)$$

$$\lesssim N^{1-\frac{q}{2}} \int_{\mathbb{D}} \Big(\sum_{|z_m|\geq\rho_0} |\lambda_m|^2 \, |f'_{z_m,n}(\xi)|^2 \Big)^{q/2} d\mu(z).$$

Therefore, for $q < 2$ and $q \geq 2$, we have

$$\sum_{|z_m|\geq\rho_0} \frac{|\lambda_m|^q}{\tau(z_m)^{2/p}} \int_{D_{3\delta}(z_m)} (1+\varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}} \, d\mu(\xi)$$

$$\lesssim \int_{\mathbb{D}} \sum_{|z_m|\geq\rho_0} |\lambda_m|^q \, |f'_{z_m,n}(\xi)|^q \chi_{D_{3\delta}(z_m)}(\xi) \, d\mu(\xi)$$

$$\lesssim \max(1, N^{1-\frac{q}{2}}) \int_{\mathbb{D}} \Big(\sum_{|z_m|\geq\rho_0} |\lambda_m|^2 \, |f'_{z_m,n}(\xi)|^2 \Big)^{q/2} d\mu(z).$$

By applying (4.7), we have

$$\sum_{|z_m|\geq\rho_0} \frac{|\lambda_m|^q}{\tau(z_m)^{2/p}} \int_{D_{3\delta}(z_m)} (1+\varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}} \, d\mu(\xi) \lesssim \|\lambda\|_{\ell^p}^q.$$

Thus, taking $|b_m| = |\lambda_m|^q \in \ell^{p/q}$ and using the duality $(\ell^p)^* = \ell^q$, we see that the sequence

$$\left\{\frac{1}{\tau(z_m)^{2/p}} \int\limits_{D_{3\delta}(z_m)} (1+\varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}}\, d\mu(\xi)\right\}_m \in \ell^{p/(p-q)}.$$

Observe that there is $\rho_0 < r_1 < 1$ such that if a point z_j of the sequence $\{z_k\}$ belongs to $\{z \in \mathbb{D} : |z| \leq \rho_0\}$, then $D_\delta(z_j) \subset \{z \in \mathbb{D} : |z| \leq r_1\}$. Thus, by Lemma C and (2.3), we get

$$\begin{aligned}
&\int\limits_{|z|\geq r_1} \left(\frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} (1+\varphi'(\xi))^q \omega(\xi)^{-q/2}\, d\mu(\xi)\right)^{p/(p-q)} dA(z) \\
&\lesssim \sum_{|z_m|\geq\rho_0} \int\limits_{D_\delta(z_m)} \left(\frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} (1+\varphi'(\xi))^q \omega(\xi)^{-q/2}\, d\mu(\xi)\right)^{p/(p-q)} dA(z) \\
&\lesssim \sum_{|z_m|\geq\rho_0} \left(\frac{1}{\tau(z_m)^{2/p}} \int\limits_{D_{3\delta}(z_m)} (1+\varphi'(\xi))^q \omega(\xi)^{-q/2}\, d\mu(\xi)\right)^{p/(p-q)} < \infty.
\end{aligned}$$

Therefore, since

$$\int\limits_{|z|\leq r_1} \left(\frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} (1+\varphi'(s))^q \omega(s)^{-\frac{q}{2}} d\mu(s)\right)^{p/p-q} d\mu(z) < \infty,$$

we obtain

$$\begin{aligned}
\int\limits_{\mathbb{D}} F_{\delta,\mu}(\varphi)(z)^{p/(p-q)}\, dA(z) &= \int\limits_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} (1+\varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}}\, d\mu(\xi)\right)^{p/(p-q)} dA(z) \\
&\lesssim \int\limits_{|z|<r_1} \left(\frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} (1+\varphi'(s))^q \omega(s)^{-\frac{q}{2}} d\mu(s)\right)^{p/p-q} dA(z) \\
&+ \int\limits_{|z|>r_1} \left(\frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} (1+\varphi'(s))^q \omega(s)^{-\frac{q}{2}} d\mu(s)\right)^{p/p-q} dA(z) < \infty.
\end{aligned}$$

This proves the desired result.

Finally, it remains to prove that (c) implies (b). Suppose that (4.6) holds and let $\{f_n\}$ be a bounded sequence of functions belonging to S^p_ω that converges uniformly to zero on compact subsets of $\mathbb{D}$. Since the function τ is decreasing and converges to zero as $|z| \to 1$, there is $r' > 0$ such that

$$D_{\delta/2}(z) \subset \Big\{\xi \in \mathbb{D} : |\xi| > r/2\Big\}, \quad \text{if } |z| > r > r'. \tag{4.8}$$

On the other hand, it also follows from (2.4) that

$$|f_n(z)|^q \frac{\omega(z)^{q/2}}{(1+\varphi'(z))^q} \lesssim \frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} |f_n(s)|^q \frac{\omega(s)^{q/2}}{(1+\varphi'(s))^q}\, dA(s).$$

Integrate with respect to $d\mu$, and use (4.8) and (2.3) to obtain

$$\begin{aligned} &\int\limits_{|z|\ge r} |f_n(\xi)|^q\, d\mu(\xi) \\ &\lesssim \int\limits_{|\xi|\ge r/2} |f_n(\xi)|^q \frac{\omega(\xi)^{q/2}}{(1+\varphi'(\xi))^q} \left(\frac{1}{\tau(\xi)^2} \int\limits_{D_\delta(\xi)} (1+\varphi'(z))^q\, \omega(z)^{-q/2}\, d\mu(z) \right) dA(\xi) \end{aligned} \tag{4.9}$$

By (c), for each $\varepsilon > 0$, there is an $r_0 > r'$ such that

$$\int\limits_{|\xi|\ge r_0/2} \left(\frac{1}{\tau(\xi)^2} \int\limits_{D_\delta(\xi)} (1+\varphi'(z))^q\, \omega(z)^{-q/2}\, d\mu(z) \right)^{p/(p-q)} dA(\xi) < \varepsilon^{p/(p-q)}.$$

Combining this with Hölder's inequality, we have

$$\begin{aligned} &\int\limits_{|z|\ge r_0} |f_n(\xi)|^q\, d\mu(\xi) \\ &\lesssim \|f_n\|^q_{S^p_\omega} \left(\int\limits_{|\xi|\ge r_0/2} \left(\frac{1}{\tau(\xi)^2} \int\limits_{D_\delta(\xi)} (1+\varphi'(z))^q\, \omega(z)^{-q/2}\, d\mu(z) \right)^{p/(p-q)} dA(\xi) \right)^{(p-q)/p} \\ &\lesssim \varepsilon. \end{aligned} \tag{4.10}$$

This together with the fact that

$$\lim_{n\to\infty} \int\limits_{|z|\le r_0} |f_n(\xi)|^q\, d\mu(\xi) = 0$$

gives $\lim_{n\to\infty} \|f_n\|_{L^q(\mu)} = 0$, which completes the proof. □

We finish this section with the case $0 < q < \infty$ and $p = \infty$.

Lemma 4.3. *Let $\omega \in \mathcal{W}$, $0 < q < \infty$, and μ be a finite positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent;*

(a) *The operator $I_\mu : S_\omega^\infty \to L^q(\mathbb{D}, d\mu)$ is bounded.*
(b) *The operator $I_\mu : S_\omega^\infty \to L^q(\mathbb{D}, d\mu)$ is compact.*
(c) *The function*

$$F_{\delta,\mu}(\varphi) \in L^1(\mathbb{D}, dA). \tag{4.11}$$

Proof. Suppose first that the operator $I_\mu : S_\omega^\infty \to L^q(\mathbb{D}, d\mu)$ is bounded. Let $\{z_m\}_m$ be a (δ, τ)-lattice on $\mathbb{D}$. Corresponding to each $\lambda = \{\lambda_m\}_m \in \ell^\infty$, we consider again

$$f(z) = \sum_{|z_m|\ge\rho_0} \lambda_m F_{z_m,n}(z),$$

where $F_{z_m,n}(z)$ is in Lemma E. By Proposition 2.4 and (1.16), we have

$$\|f'\|_{S_\omega^\infty} \le \|f\|_{A_\omega^\infty} \lesssim \|\lambda\|_{\ell^\infty}.$$

By our assumption, we get

$$\int_{\mathbb{D}} \Big| \sum_{|z_m|\ge\rho_0} \lambda_m F'_{z_m,n}(z)\Big|^q \omega(z)^{q/2}\, d\mu(z) \lesssim \|\lambda\|_{\ell^\infty}^q$$

and so

$$\int_{\mathbb{D}} \left(\sum_{|z_m|\ge\rho_0} |\lambda_m|^2 |F'_{z_m,n}(z)|^2\, \omega(z) \right)^{q/2} d\mu(z) \lesssim \|\lambda\|_{\ell^\infty}^q.$$

This together with Lemma C, Lemma 2.3 and Hölder's inequality imply that

$$\begin{aligned}\sum_{|z_m|\ge\rho_0} |\lambda_m|^q \int_{D_{3\delta}(z_m)} & (1+\varphi'(\xi))^q \omega(\xi)^{-\frac{q}{2}}\, d\mu(\xi) \\ &\lesssim \max(1, N^{1-\frac{q}{2}}) \int_{\mathbb{D}} \Big(\sum_{|z_m|\ge\rho_0} |\lambda_m|^2\, |F'_{z_m,n}(\xi)|^2\, \omega(\xi)\Big)^{q/2} d\mu(z) \lesssim \|\lambda\|_{\ell^\infty}^q.\end{aligned}$$

Then, taking $|\lambda_m| = 1$ gives

$$\sum_{|z_m|\geq\rho_0}\int\limits_{D_{3\delta}(z_m)}(1+\varphi'(\xi))^q\omega(\xi)^{-\frac{q}{2}}\,d\mu(\xi)\lesssim 1. \tag{4.12}$$

As in the previous proof, by Lemma C, (2.3) and (4.12), we get

$$\begin{aligned}
&\int\limits_{|z|\geq r_1}\left(\frac{1}{\tau(z)^2}\int\limits_{D_\delta(z)}(1+\varphi'(\xi))^q\omega(\xi)^{-q/2}\,d\mu(\xi)\right)dA(z)\\
&\lesssim\sum_{|z_m|\geq\rho_0}\int\limits_{D_\delta(z_m)}\left(\frac{1}{\tau(z)^2}\int\limits_{D_\delta(z)}(1+\varphi'(\xi))^q\omega(\xi)^{-q/2}\,d\mu(\xi)\right)dA(z)\\
&\lesssim\sum_{|z_m|\geq\rho_0}\int\limits_{D_{3\delta}(z_m)}(1+\varphi'(\xi))^q\omega(\xi)^{-q/2}\,d\mu(\xi)<\infty.
\end{aligned}$$

Combining this with the fact that

$$\int\limits_{|z|\leq r_1}\left(\frac{1}{\tau(z)^2}\int\limits_{D_\delta(z)}(1+\varphi'(\xi))^q\omega(\xi)^{-q/2}\,d\mu(\xi)\right)dA(z)<\infty,$$

we have the desired result—see (4.5).

It remains to show that (c) implies (b). Let $\{f_n\}$ be a bounded sequence of functions in S^∞_ω converging uniformly to zero on compact subsets of $\mathbb{D}$. Since the function $\tau(z)$ is decreasing and converges to zero as $|z|\to 1$, there is $r'>0$ such that

$$D_{\delta/2}(z)\subset\Big\{\xi\in\mathbb{D}:|\xi|>r/2\Big\},\qquad \text{if } |z|>r>r'. \tag{4.13}$$

On the other hand, it follows from (2.4) that

$$|f_n(z)|^q\,\frac{\omega(z)^{q/2}}{(1+\varphi'(z))^q}\lesssim\frac{1}{\tau(z)^2}\int\limits_{D_\delta(z)}|f_n(s)|^q\,\frac{\omega(s)^{q/2}}{(1+\varphi'(s))^q}\,dA(s).$$

Integrate with respect to $d\mu$, and use (4.13), (2.3), and (2.4), to obtain

$$\begin{aligned}
&\int\limits_{|z|\geq r}|f_n(\xi)|^q\,d\mu(\xi)\\
&\lesssim\int\limits_{|\xi|\geq r/2}|f_n(\xi)|^q\,\frac{\omega(\xi)^{q/2}}{(1+\varphi'(\xi))^q}\left(\frac{1}{\tau(\xi)^2}\int\limits_{D_\delta(\xi)}(1+\varphi'(z))^q\,\omega(z)^{-q/2}\,d\mu(z)\right)dA(\xi)
\end{aligned}$$

$$\lesssim \|f_n\|_{S_\omega^\infty}^q \int\limits_{|\xi|\geq r/2} \frac{1}{\tau(\xi)^2} \int\limits_{D_\delta(\xi)} (1+\varphi'(z))^q \, \omega(z)^{-q/2} \, d\mu(z) dA(\xi). \tag{4.14}$$

Now the rest follows as in the previous proof. □

5. Proofs of Theorems 1.1 and 1.2

5.1. Proof of Theorem 1.1 (A)

Let $0 < p \leq q < \infty$. By (1.15),

$$\|GI_{(\phi,g)} f(z)\|_{A_\omega^q}^q \asymp \int\limits_{\mathbb{D}} |f'(\phi(z))|^q \, |g(z)|^q \, \frac{\omega(z)^{q/2}}{(1+\varphi'(z))^q} \, dA(z)$$

$$= \int\limits_{\mathbb{D}} |f'(z)|^q \, d\mu_{\phi,\omega,q}(z) = \|f'\|_{L^q(\mathbb{D}, d\mu_{\phi,\omega,g})}^q.$$

Therefore, $GI_{(\phi,g)} : A_\omega^p \to A_\omega^q$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S_\omega^p \to L^q(\mu_{\phi,\omega,g})$ is bounded. Using (a) of Lemma 4.1, this is equivalent to

$$\sup_{z\in\mathbb{D}} \frac{1}{\tau(z)^{2q/p}} \int\limits_{D_\delta(z)} (1+\varphi'(\xi))^q \, \omega(\xi)^{-q/2} \, d\mu_{\phi,\omega,g}(\xi) < \infty,$$

which, by Theorem B, is equivalent to

$$\sup_{z\in\mathbb{D}} \tau(z)^{2(1-q/p)} \int\limits_{\mathbb{D}} |k_{q,z}(\xi)|^q \omega(\xi)^{q/2} \, d\nu_{\phi,\omega,g}(\xi) < \infty.$$

Now, by (a) of Lemma 2.2, we get

$$\tau(z)^{2(1-q/p)} \int\limits_{\mathbb{D}} |k_{q,z}(\xi)|^q \, \omega(\xi)^{q/2} \, dv_{\phi,\omega,q}(\xi)$$

$$\asymp \int\limits_{\mathbb{D}} |k_{p,z}(\xi)|^q \, \omega(\xi)^{q/2} \, dv_{\phi,\omega,q}(\xi)$$

$$= \int\limits_{\mathbb{D}} |k_{p,z}(\xi)|^q \, (1+\varphi'(\phi(\xi))^q \, d\mu_{\phi,\omega,q}(\xi)$$

$$= \int\limits_{\mathbb{D}} |k_{p,z}(\xi)|^q \, |g(z)|^q \, \frac{(1+\varphi'(\phi(\xi)))^q}{(1+\varphi'(\xi))^q} \, \omega(z)^{q/2} \, dA(\xi)$$

$$= GB_{1,p,q}^\phi. \tag{5.1}$$

Thus, $GI_{\phi,g}$ is bounded if and only if $GB^{\phi}_{1,p,q}(g(z)) \in L^{\infty}(\mathbb{D}, dA)$. Compactness can be proved similarly using (b) of Lemma 4.1.

5.2. Proof of Theorem 1.1 (B)

Boundedness. Let $0 < p < q = \infty$ and suppose first that (1.4) holds. Then by (1.16) and our assumption, we have

$$\begin{aligned}\|GI_{\phi,g}f\|_{A^{\infty}_{\omega}} &\asymp \sup_{z\in\mathbb{D}} |f'(\phi(z))|\,|g(z)|\,\frac{\omega(z)^{1/2}}{(1+\varphi'(z))}\\ &\le \sup_{z\in\mathbb{D}} M_{g,\phi,\omega}(z)\,\sup_{z\in\mathbb{D}} \frac{|f'(\phi(z))|\,\omega(\phi(z))^{\frac{1}{2}}}{(1+\varphi'(\phi(z)))}\,\Delta\varphi(\phi(z))^{-1/p}\\ &\le \sup_{z\in\mathbb{D}} M_{g,\phi,\omega}(z)\,\sup_{z\in\mathbb{D}} \frac{|f'(\phi(z))|\,\omega(\phi(z))^{\frac{1}{2}}}{(1+\varphi'(\phi(z)))}\,\tau(\phi(z))^{2/p}.\end{aligned}$$

Therefore, by (2.4),

$$\begin{aligned}\|GI_{(\phi,g)}f\|_{A^{\infty}_{\omega}} &\lesssim \sup_{z\in\mathbb{D}}\left(\int\limits_{D_{\delta}(\phi(z))} \frac{|f'(\xi)|^{p}\omega(\xi)^{\frac{p}{2}}}{(1+\varphi'(\xi))^{p}}\,dA(\xi)\right)^{1/p}\\ &\le \left(\int\limits_{\mathbb{D}} \frac{|f'(\xi)|^{p}\omega(\xi)^{\frac{p}{2}}}{(1+\varphi'(\xi))^{p}}\,dA(\xi)\right)^{1/p} \lesssim \|f\|_{A^{p}_{\omega}},\end{aligned}$$

which implies that $GI_{(\phi,g)} : A^p_{\omega} \to A^q_{\omega}$ is bounded.

Conversely, suppose that the operator $GI_{(\phi,g)} : A^p_{\omega} \to A^{\infty}_{\omega}$ is bounded. Choose $\xi \in \mathbb{D}$ so that $|\phi(\xi)| > \rho_0$, and consider the function $f_{\phi(\xi),n,p}$ given by

$$f_{\phi(\xi),n,p} := \frac{F_{\phi(\xi),n,p}}{\tau(\phi(\xi))^{2/p}},$$

where $F_{\phi(\xi),n,p}$ is the test function in Lemma E. Notice that $f_{\phi(\xi),n,p}$ is in A^p_{ω} and $\|f_{\phi(\xi),n,p}\|_{A^p_{\omega}} \asymp 1$. By our assumption, we get

$$\begin{aligned}\|GI_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^{\infty}_{\omega}} &\ge \sup_{z\in\mathbb{D}} \frac{|f'_{\phi(\xi),n,p}(z)||g(z)|}{(1+\varphi'(z))}\,\omega(z)^{\frac{1}{2}}\\ &\ge \sup_{z\in\mathbb{D}} \frac{|F'_{\phi(\xi),n,p}(z)||g(z)|}{\tau(\phi(\xi))^{2/p}(1+\varphi'(z))}\,\omega(z)^{\frac{1}{2}}\\ &\ge \sup_{\xi\in\mathbb{D}} \frac{|F'_{\phi(\xi),n,p}(\phi(\xi))||g(\xi)|}{\tau(\phi(\xi))^{2/p}(1+\varphi'(\xi))}\,\omega(\xi)^{\frac{1}{2}}.\end{aligned}$$

By Lemma 2.3,

$$|F'_{\phi(\xi),n,p}(z)|\,\omega(z)^{1/2}\asymp(1+\varphi'(z)),\ z\in D_\delta(\phi(\xi)),$$

and so

$$\begin{aligned}\infty>\|GI_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^\infty_\omega}&\geq|g(\xi)|\,\frac{(1+\varphi'(\phi(\xi)))}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac12}}{\omega(\phi(\xi))^{\frac12}}\,\tau(\phi(\xi))^{-2/p}\\&\asymp|g(\xi)|\,\frac{(1+\varphi'(\phi(\xi)))}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac12}}{\omega(\phi(\xi))^{\frac12}}\,\Delta\varphi(\phi(\xi))^{1/p}\\&=M_{g,\phi,\omega}(\xi).\end{aligned}\tag{5.2}$$

On the other hand, by taking $f(z)=z$ and using the boundedness of the operator $GI_{(\phi,g)}:A^p_\omega\to A^\infty_\omega$, we obtain

$$\|GI_{(\phi,g)}f\|_{A^\infty_\omega}=\sup_{z\in\mathbb{D}}\frac{|g(z)|}{(1+\varphi'(z))}\,\omega(z)^{\frac12}\lesssim\|f\|_{A^p_\omega}<\infty.$$

Therefore, in the case of $|\phi(\xi)|\leq\rho_0$, $\xi\in\mathbb{D}$, we have

$$\begin{aligned}|M_{g,\phi,\omega}(\xi)|&=|g(\xi)|\,\frac{(1+\varphi'(\phi(\xi)))}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac12}}{\omega(\phi(\xi))^{\frac12}}\,\Delta\varphi(\phi(\xi))^{1/p}\\&\asymp|g(\xi)|\,\frac{(1+\varphi'(\phi(\xi)))}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac12}}{\omega(\phi(\xi))^{\frac12}}\,\tau(\phi(\xi))^{-2/p}\\&\leq C_1\,\frac{|g(\xi)|}{(1+\varphi'(\xi))}\,\omega(\xi)^{\frac12}<\infty,\end{aligned}$$

where

$$C_1=\sup_{|\phi(\xi)|\leq\rho_0}\left\{(1+\varphi'(\phi(\xi)))\omega(\phi(\xi))^{\frac{-1}{2}}\,\tau(\phi(\xi))^{-2/p}\right\}<\infty.$$

Combining this with (5.2) completes the proof of boundedness.

Compactness. Suppose now that $GI_{(\phi,g)}:A^p_\omega\to A^\infty_\omega$ is compact. Then, since $f_{\phi(\xi),n,p}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|\phi(\xi)|\to1$ (see Lemma 3.1 of [15]), it follows that

$$\|GI_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^\infty_\omega}\to0$$

as $|\phi(\xi)|\to1$. Thus, by (5.2),

$$0=\lim_{|\phi(\xi)|\to1^-}\|GI_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^\infty_\omega}\gtrsim\lim_{|\phi(\xi)|\to1^-}M_{g,\phi,\omega}(\xi).$$

To prove the converse, let $\{f_n\}$ be a bounded sequence of functions in A^p_ω converging uniformly to zero on compact subsets of $\mathbb{D}$. Since (1.4) holds, for each $\varepsilon > 0$, there exists an $r_0 > 0$ such that

$$M_{g,\phi,\omega}(\xi) = |g(\xi)| \frac{1+\varphi'(\phi(\xi)}{1+\varphi'(\xi)} \frac{\omega(\xi)^{1/2}}{\omega(\phi(\xi))^{1/2}} \Delta\varphi(\phi(\xi))^{1/p} < \varepsilon,$$

whenever $|\phi(\xi)| > r_0$. In addition, by (2.4),

$$\begin{aligned} &\frac{|f_n'(\phi(\xi))||g(\xi)|}{1+\varphi'(\xi)}\, \omega(\xi)^{\frac{1}{2}} \\ &\lesssim \Big(\frac{1}{\tau(z)^2} \int\limits_{D_\delta(z)} \frac{|f_n'(\phi(s))|^p}{(1+\varphi'(\phi(s)))^p}\, \omega(\phi(s))^{\frac{p}{2}}\, \Delta\varphi(\phi(\xi))\, dA(s)\Big)^{1/p} M_{g,\phi,\omega}(\xi) \qquad (5.3) \\ &\lesssim \|f_n\|_{A^p_\omega} M_{g,\phi,\omega}(\xi) < \varepsilon, \end{aligned}$$

whenever $|\phi(\xi)| > r_0$.

For $|\phi(\xi)| \geq r_0$, we have

$$\sup_{|\phi(\xi)|\leq r_0} \frac{|f_n'(\phi(\xi))||g(\xi)|}{1+\varphi'(\xi)}\, \omega(\xi)^{\frac{1}{2}} \lesssim \sup_{|\phi(\xi)|\leq r_0} |f_n'(\phi(\xi))| \to 0,$$

as $n \to \infty$ because the sequence of functions f_n' also converges uniformly to zero on compact subsets of $\mathbb{D}$ (see Lemma B). This together with (5.3) yields

$$\|GI_{(\phi,g)}(f_n)\|_{A^\infty_\omega} \asymp \sup_{\xi\in\mathbb{D}} \frac{|f_n'(\phi(z))||g(\xi)|}{1+\varphi'(\xi)}\, \omega(\xi)^{\frac{1}{2}} \to 0, \quad n \to \infty,$$

which shows the compactness of the operator $GI_{(\phi,g)} : A^p_\omega \to A^\infty_\omega$.

5.3. Proof of Theorem 1.1 (C)

Boundedness. Let $p = q = \infty$ and suppose that (1.5) holds. Using (1.16), we get

$$\begin{aligned} \|GI_{(\phi,g)} f\|_{A^\infty_\omega} &\asymp \sup_{z\in\mathbb{D}} \frac{|f'(\phi(z))||g(z)|}{(1+\varphi'(z))}\, \omega(z)^{\frac{1}{2}} \\ &\leq \sup_{z\in\mathbb{D}} N_{g,\phi,\omega}(z) \sup_{z\in\mathbb{D}} \frac{|f'(\phi(z))|\omega(\phi(z))^{\frac{1}{2}}}{(1+\varphi'(\phi(z)))} \\ &\leq \sup_{z\in\mathbb{D}} N_{g,\phi,\omega}(z) \sup_{z\in\mathbb{D}} \frac{|f'(z)|\omega(z)^{\frac{1}{2}}}{(1+\varphi'(z))} \lesssim \|f\|_{A^\infty_\omega} \end{aligned}$$

which shows that $GI_{(\phi,g)}$ is bounded.

Conversely, suppose that $GI_{(\phi,g)} : A^\infty_\omega \to A^\infty_\omega$ is bounded. Let $\xi \in \mathbb{D}$ be such that $|\phi(\xi)| > \rho_0$. Then $F_{\phi(\xi),n,p} \in A^\infty_\omega$ and $\|F_{\phi(\xi),n,p}\|_{A^\infty_\omega} \asymp 1$, and hence

$$\infty > \|GI_{(\phi,g)}(F_{\phi(\xi),n,p})\|_{A^\infty_\omega} = \sup_{z\in\mathbb{D}} \frac{|F'_{\phi(\xi),n,p}(z)||g(z)|}{(1+\varphi'(z))}\,\omega(z)^{\frac{1}{2}}$$
$$\geq \frac{|F'_{\phi(\xi),n,p}(\phi(\xi))||g(\xi)|}{(1+\varphi'(\xi))}\,\omega(\xi)^{\frac{1}{2}}.$$

By Lemma 2.3,

$$|F'_{\phi(\xi),n,p}(z)|\,\omega(z)^{1/2} \asymp (1+\varphi'(z)),\ z\in D_\delta(\phi(\xi)),$$

so

$$\infty > \|GI_{(\phi,g)}(F_{\phi(\xi),n,p})\|_{A^\infty_\omega} \asymp |g(\xi)|\,\frac{(1+\varphi'(\phi(\xi)))}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}} = NI_{g,\phi,\omega}(\xi). \tag{5.4}$$

To deal with the case $|\phi(\xi)| \leq \rho_0$, take $f(z) = z$ and use the boundedness of the operator $GI_{(\phi,g)}$ to obtain

$$\|GI_{(\phi,g)}f\|_{A^\infty_\omega} = \sup_{z\in\mathbb{D}} \frac{|g(z)|}{(1+\varphi'(z))}\,\omega(z)^{\frac{1}{2}} \lesssim \|f\|_{A^\infty_\omega} < \infty. \tag{5.5}$$

Therefore, when $|\phi(\xi)| \leq \rho_0$, $\xi\in\mathbb{D}$, we have

$$|g(\xi)|\,\frac{(1+\varphi'(\phi(\xi)))}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}} \leq C_2\,\frac{|g(\xi)|}{(1+\varphi'(\xi))}\,\omega(\xi)^{\frac{1}{2}} < \infty,$$

where

$$C_2 = \sup_{|\phi(\xi)|\leq\rho_0} \left\{(1+\varphi'(\phi(\xi)))\omega(\phi(\xi))^{\frac{-1}{2}}\right\} < \infty.$$

Combining this with (5.4) completes the proof of boundedness.

Compactness. If $GI_{(\phi,g)} : A^\infty_\omega \to A^\infty_\omega$ is compact, then, using (5.4) again, we get

$$\lim_{|\phi(\xi)|\to 1^-} NI_{g,\phi,\omega}(\xi) \lesssim \lim_{|\phi(\xi)|\to 1^-} \|GI_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^\infty_\omega} = 0.$$

To prove the converse, let $\{f_n\}$ be a bounded sequence of functions in A^∞_ω converging uniformly to zero on compact subsets of $\mathbb{D}$. By assumption, for any $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$NI_{g,\phi,\omega}(\xi) = |g(z)|\,\frac{(1+\varphi'(\phi(z))}{(1+\varphi'(z))}\,\frac{\omega(z)^{1/2}}{\omega(\phi(z))^{1/2}} < \varepsilon,$$

whenever $|\phi(\xi)| > r_0$. Notice that

$$\begin{aligned}\frac{|f_n'(\phi(\xi))||g(\xi)|}{1+\varphi'(\xi)}\,\omega(\xi)^{\frac{1}{2}} &\lesssim \|f_n\|_{A^\infty_\omega}\,|g(\xi)|\,\frac{(1+\varphi'(\phi(\xi))}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{1/2}}{\omega(\phi(\xi))^{1/2}} \\ &= \|f_n\|_{A^\infty_\omega}\,N_{g,\phi,\omega}(\xi) < \varepsilon,\end{aligned} \tag{5.6}$$

whenever $|\phi(\xi)| > r_0$. The rest follows as in the proof of (B).

5.4. Proof of Theorem 1.1 (D)

Let $0 < q < p < \infty$ and suppose that $GI_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded. If $\{f_n\} \subset A^p_\omega$ is a bounded sequence converging to zero uniformly on compact subsets of $\mathbb{D}$, then

$$\|GI_{(\phi,g)} f_n\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} \frac{|f_n'(\phi(z))|^q |g(z)|^q}{(1+\varphi'(z))^q}\,\omega(z)^{\frac{q}{2}}\,dA(z) = \|f_n'\|^q_{L^q(\mu_{\phi,\omega,g})}, \tag{5.7}$$

which goes to zero as $n \to \infty$ because of the compactness of the embedding $I_{\mu_{\phi,\omega,g}}$.

We next prove that (a) and (c) are equivalent. By (5.7) and Lemma 4.2, we get $GI_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S^p_\omega \to L^q(\mu_{\phi,\omega,g})$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S^p_\omega \to L^q(\mu_{\phi,\omega,g})$ is compact if and only if the function

$$F_{\delta,\mu_{\phi,\omega,g}}(\varphi)(z) := \frac{1}{\tau(z)^2} \int_{D_\delta(z)} (1+\varphi'(\xi))^q \omega(\xi)^{-q/2}\,d\mu_{\phi,\omega,g}(\xi)$$

belongs to $L^{p/(p-q)}(\mathbb{D}, dA)$. By Theorem C, this is equivalent to

$$\int_{\mathbb{D}} |k_{q,z}(\xi)|^q\,\omega(\xi)^{q/2}\,d\nu_{\phi,\omega,g}(\xi) \in L^{p/(p-q)}(\mathbb{D}, dA),$$

which is as well equivalent to $GB^{\phi}_{1,p,q}(g)(z) \in L^{p/(p-q)}(\mathbb{D}, d\lambda)$, where $d\lambda(z) = dA(z)/\tau(z)^2$, because of

$$\begin{aligned}&\int_{\mathbb{D}} G_q(\nu^q_{\phi,\omega,g})^{p/p-q} dA(z) \\ &= \int_{\mathbb{D}} \Big(\tau(z)^{2(1-q/p)}\,G_q(\nu^q_{\phi,\omega,g})\Big)^{\frac{p}{p-q}} d\lambda(z) \\ &\asymp \int_{\mathbb{D}} \Big(\tau(z)^{2(1-q/p)} \int_{\mathbb{D}} |k_{p,z}(\xi)|^q \omega(\xi)^{q/2}\,d\nu_{\phi,\omega,g}(\xi)\Big)^{\frac{p}{p-q}} d\lambda(z) \\ &= \int_{\mathbb{D}} \Big(\tau(z)^{2(1-q/p)} \int_{\mathbb{D}} |k_{p,z}(\xi)|^q\,(1+\varphi'(\xi))^q\,d\mu_{\phi,\omega,g}(\xi)\Big)^{\frac{p}{p-q}} d\lambda(z) \\ &= \int_{\mathbb{D}} GB^{\phi}_{1,p,q}(g)(z)^{p/p-q}\,d\lambda(z).\end{aligned}$$

This completes the proof of (D) when $0 < q < p < \infty$.

Suppose that $0 < q < p = \infty$. If $GI_{(\phi,g)} : A^\infty_\omega \to A^q_\omega$ is bounded and $\{f_n\} \subset A^\infty_\omega$ is a bounded sequence converging to zero uniformly on compact subsets of $\mathbb{D}$, then

$$\|GI_{(\phi,g)} f_n\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} \frac{|f_n'(\phi(z))|^q |g(z)|^q}{(1+\varphi'(z))^q}\, \omega(z)^{\frac{q}{2}}\, dA(z) = \|f_n'\|^q_{L^q(\mu_{\phi,\omega,g})} \to 0, \tag{5.8}$$

where we used again the compactness of the embedding $I_{\mu_{\phi,\omega,g}}$, and so $GI_{(\phi,g)}$ is compact.

It remains to prove that (a) and (c) are equivalent. By (5.8) and Lemma 4.3, we get $GI_{(\phi,g)} : A^\infty_\omega \to A^q\omega$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S^\infty_\omega \to L^q(\mu_{\phi,\omega,g})$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S^\infty_\omega \to L^q(\mu_{\phi,\omega,g})$ is compact if and only if the function

$$F_{\delta,\mu_{\phi,\omega,g}}(\varphi)(z) := \frac{1}{\tau(z)^{2q/p}} \int_{D_\delta(z)} (1+\varphi'(\xi))^q \omega(\xi)^{-q/2}\, d\mu_{\phi,\omega,g}(\xi)$$

belongs to $L^1(\mathbb{D}, dA)$. By Theorem E, this is equivalent to

$$\int_{\mathbb{D}} |k_{q,z}(\xi)|^q\, \omega(\xi)^{-q/2}\, d\nu_{\phi,\omega,g}(\xi) \in L^1(\mathbb{D}, dA),$$

which is in turn equivalent to $GB^\phi_{1,p,q}(g)(z) \in L^1(\mathbb{D}, d\lambda)$, where $d\lambda(z) = dA(z)/\tau(z)^2$, because of

$$GB^\phi_{1,p,q}(g)(z) \asymp \tau(z)^{2(1-q/p)} \int_{\mathbb{D}} |k_{q,z}(\xi)|^q \omega(\xi)^{q/2}\, d\nu^q_{\phi,\omega,g}(\xi).$$

5.5. *Proof of Theorem 1.2 (A)*

Boundedness. Let $0 < p \le q < \infty$. By (1.15),

$$\|GV_{(\phi,g)} f\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} \frac{|f(\phi(z))|^q |g(z)|^q}{(1+\varphi'(z))^q}\, \omega(z)^{\frac{q}{2}}\, dA(z) = \int_{\mathbb{D}} |f(z)|^q\, \omega(z)^{\frac{q}{2}}\, d\nu_{\phi,\omega,g}. \tag{5.9}$$

Therefore, $GV_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded if and only if the measure $\nu_{\phi,\omega,g}$ is a q-Carleson measure for A^p_ω. According to Theorem B, this is equivalent to

$$\sup_{z\in\mathbb{D}} \tau(z)^{2(1-q/p)} \int_{\mathbb{D}} |k_{q,z}(\xi)|^q \omega(\xi)^{q/2}\, d\nu_{\phi,\omega,g}(\xi) < \infty.$$

Now, using (a) of Lemma 2.2, we get

$$\begin{aligned}\tau(z)^{2(1-q/p)}\int_{\mathbb{D}}|k_{q,z}(\xi)|^q\,\omega(\xi)^{q/2}\,d\nu_{\phi,\omega,q}(\xi) &\asymp \int_{\mathbb{D}}|k_{p,z}(\xi)|^q\,\omega(\xi)^{q/2}\,d\nu_{\phi,\omega,q}(\xi)\\ &= \int_{\mathbb{D}}|k_{p,z}(\phi(\xi))|^q\,\frac{|g(z)|^q}{(1+\varphi'(\xi))^q}\,\omega(\xi)^{q/2}dA(\xi)\\ &= GB^{\phi}_{0,p,q}.\end{aligned}$$

Thus, $GV_{(\phi,g)}$ is bounded if and only if $GB^{\phi}_{0,p,q}(g)\in L^\infty(\mathbb{D},dA)$.

Compactness. By above, $GV_{(\phi,g)}:A^p_\omega\to A^q_\omega$ is compact if and only if the measure $\nu_{\phi,\omega,g}$ is a vanishing q-Carleson measure for A^p_ω. This is equivalent to

$$\lim_{|z|\to 1^-}\tau(z)^{2(1-q/p)}\int_{\mathbb{D}}|k_{q,z}(\xi)|^q\omega(\xi)^{q/2}\,d\nu_{\phi,\omega,g}(\xi)=0.$$

Now, using (a) of Lemma 2.2, we get

$$\begin{aligned}&\tau(z)^{2(1-q/p)}\int_{\mathbb{D}}|k_{q,z}(\xi)|^q\,\omega(\xi)^{q/2}\,d\nu_{\phi,\omega,q}(\xi)\\ &\asymp \int_{\mathbb{D}}|k_{p,z}(\xi)|^q\,\omega(\xi)^{q/2}\,d\nu_{\phi,\omega,q}(\xi)\\ &= \int_{\mathbb{D}}|k_{p,z}(\phi(\xi))|^q\,\frac{|g(z)|^q}{(1+\varphi'(\xi))^q}\,\omega(z)^{q/2}\,dA(\xi)\\ &= GB^{\phi}_{0,p,q}.\end{aligned}$$

Therefore, $\lim_{|z|\to 1^-}GB^{\phi}_{0,p,q}(g)=0$ if and only if $GV_{(\phi,g)}$ is compact.

5.6. Proof of Theorem 1.2 (B)

Boundedness. Let $0<p<q=\infty$ and suppose that (1.6) holds. Then, by (1.16), we have

$$\begin{aligned}\|GV_{(\phi,g)}f\|_{A^\infty_\omega} &\asymp \sup_{z\in\mathbb{D}}\frac{|f(\phi(z))||g(z)|}{(1+\varphi'(z))}\,\omega(z)^{\frac{1}{2}}\\ &\le \sup_{z\in\mathbb{D}}MV_{g,\phi,\omega}(z)\,\sup_{z\in\mathbb{D}}|f(\phi(z))|\omega(\phi(z))^{\frac{1}{2}}\,\Delta\varphi(\phi(z))^{-1/p}\\ &\lesssim \sup_{z\in\mathbb{D}}MV_{g,\phi,\omega}(z)\,\sup_{z\in\mathbb{D}}|f(\phi(z))|\omega(\phi(z))^{\frac{1}{2}}\,\tau(\phi(z))^{2/p}.\end{aligned}$$

By (2.4) for $f\in A^p_\omega$, we obtain

$$\|GV_{(\phi,g)}f\|_{A^\infty_\omega} \lesssim \sup_{z\in\mathbb{D}} \left(\int\limits_{D_\delta(\phi(z))} |f(\xi)|^p \omega(\xi)^{\frac{p}{2}}\, dA(\xi) \right)^{1/p}$$
$$\leq \left(\int\limits_{\mathbb{D}} |f(\xi)|^p \omega(\xi)^{\frac{p}{2}}\, dA(\xi) \right)^{1/p} = \|f\|_{A^p_\omega}.$$

Therefore, the operator $GV_{(\phi,g)}$ is bounded.

Conversely, suppose that the operator $GV_{(\phi,g)} : A^p_\omega \to A^\infty_\omega$ is bounded. Taking $\xi \in \mathbb{D}$ such that $|\phi(\xi)| > \rho_0$, we consider the function $f_{\phi(\xi),n,p}$ given by $f_{\phi(\xi),n,p} := \frac{F_{\phi(\xi),n,p}}{\tau(\phi(\xi))^{2/p}}$ where $F_{\phi(\xi),n,p}$ is the test function defined in Lemma E. These functions $f_{\phi(\xi),n,p}$ belong to A^p_ω with $\|f_{\phi(\xi),n,p}\|_{A^p_\omega} \asymp 1$. By (1.16),

$$\infty > \|GV_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^\infty_\omega} \geq \sup_{z\in\mathbb{D}} \frac{|f_{\phi(\xi),n,p}(z)||g(z)|}{(1+\varphi'(z))}\, \omega(z)^{\frac{1}{2}}$$
$$\geq \sup_{z\in\mathbb{D}} \frac{|F_{\phi(\xi),n,p}(z)||g(z)|}{\tau(\phi(\xi))^{2/p}(1+\varphi'(z))}\, \omega(z)^{\frac{1}{2}}$$
$$\geq \frac{|F_{\phi(\xi),n,p}(\phi(\xi))||g(\xi)|}{\tau(\phi(\xi))^{2/p}(1+\varphi'(\xi))}\, \omega(\xi)^{\frac{1}{2}}\, \frac{\omega(\phi(\xi))^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}}.$$

In this case, by (2.11),

$$\infty > \|GV_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^\infty_\omega} \geq \frac{|g(\xi)|}{(1+\varphi'(\xi))}\, \frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}}\, \tau(\phi(\xi))^{-2/p}$$
$$\asymp \frac{|g(\xi)|}{(1+\varphi'(\xi))}\, \frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}}\, \Delta\varphi(\phi(\xi))^{1/p} = MV_{g,\phi,\omega}(\xi). \tag{5.10}$$

On the other hand, if we define $f(z) = z$ and use the boundedness of the operator $GV_{(\phi,g)} : A^p_\omega \to A^\infty_\omega$, we obtain

$$\|GV_{(\phi,g)}f\|_{A^\infty_\omega} \asymp \sup_{z\in\mathbb{D}} \frac{|g(z)|}{(1+\varphi'(z))}\, \omega(z)^{\frac{1}{2}} \lesssim \|f\|_{A^p_\omega} < \infty. \tag{5.11}$$

Therefore, in the case of $|\phi(\xi)| \leq \rho_0$, $\xi \in \mathbb{D}$, we have

$$\frac{|g(\xi)|}{(1+\varphi'(\xi))}\, \frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}}\, \Delta\varphi(\phi(\xi))^{1/p} \asymp \frac{|g(\xi)|}{(1+\varphi'(\xi))}\, \frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}}\, \tau(\phi(\xi))^{-2/p}$$
$$\leq C_1 \frac{|g(\xi)|}{(1+\varphi'(\xi))}\, \omega(\xi)^{\frac{1}{2}} < \infty,$$

where

$$C_1 = \sup_{|\phi(\xi)| \le \rho_0} \left\{ \omega(\phi(\xi))^{\frac{-1}{2}} \tau(\phi(\xi))^{-2/p} \right\} < \infty.$$

This, (1.6) holds.

Compactness. Suppose that the operator $GV_{(\phi,g)} : A^p_\omega \to A^\infty_\omega$ is compact and define

$$f_{\phi(\xi),n,p} := \frac{F_{\phi(\xi),n,p}}{\tau(\phi(\xi))^{2/p}}, \quad \text{for } |\phi(\xi)| > \rho_0,$$

which are in A^p_ω and converge uniformly to zero on compact subsets of $\mathbb{D}$ as $|\phi(\xi)| \to 1$. Thus,

$$\|GI_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^\infty_\omega} \to 0$$

as $|\phi(\xi)| \to 1$. Thus, (5.10) shows that

$$\lim_{|\phi(\xi)| \to 1^-} MV_{g,\phi,\omega}(\xi) \lesssim \lim_{|\phi(\xi)| \to 1^-} \|GV_{(\phi,g)}(f_{\phi(\xi),n,p})\|_{A^\infty_\omega} = 0.$$

Conversely, if $\{f_n\}$ is a bounded sequence of functions in A^p_ω converging uniformly to zero on compact subsets of $\mathbb{D}$, then, as for $GI_{(\phi,g)}$, it follows that

$$\|GV_{(\phi,g)}(f_n)\|_{A^\infty_\omega} = \sup_{\xi \in \mathbb{D}} \frac{|f_n(\phi(\xi))||g(\xi)|}{1 + \varphi'(\xi)} \omega(\xi)^{\frac{1}{2}} \to 0, \quad n \to \infty,$$

which proves the compactness of the operator $GV_{(\phi,g)} : A^p_\omega \to A^\infty_\omega$.

5.7. *Proof of Theorem 1.2 (C)*

Boundedness. Let $p = q = \infty$. Suppose first that (1.7) holds. Then, by (1.16),

$$\begin{aligned}
\|GV_{(\phi,g)} f\|_{A^\infty_\omega} &= \sup_{z \in \mathbb{D}} \frac{|f(\phi(z))||g(z)|}{(1 + \varphi'(z))} \omega(z)^{\frac{1}{2}} \\
&\le \sup_{z \in \mathbb{D}} NV_{g,\phi,\omega}(z) \sup_{z \in \mathbb{D}} |f(\phi(z))| \omega(\phi(z))^{\frac{1}{2}} \\
&\le \sup_{z \in \mathbb{D}} NV_{g,\phi,\omega}(z) \sup_{z \in \mathbb{D}} |f(z)| \omega(z)^{\frac{1}{2}} \lesssim \|f\|_{A^\infty_\omega},
\end{aligned}$$

that is, $GV_{(\phi,g)}$ bounded.

Conversely, suppose that $GV_{(\phi,g)} : A^\infty_\omega \to A^\infty_\omega$ is bounded and show that (1.7) holds. As before, if $\xi \in \mathbb{D}$ is such that $|\phi(\xi)| > \rho_0$, we use the test functions $F_{\phi(\xi),n,p}$ to obtain

$$\begin{aligned}\infty > \|GV_{(\phi,g)}(F_{\phi(\xi),n,p})\|_{A^\infty_\omega} &= \sup_{z\in\mathbb{D}} \frac{|F_{\phi(\xi),n,p}(z)||g(z)|}{(1+\varphi'(z))}\,\omega(z)^{\frac{1}{2}} \\ &\geq \frac{|F_{\phi(\xi),n,p}(\phi(\xi))||g(\xi)|}{(1+\varphi'(\xi))}\,\omega(\xi)^{\frac{1}{2}}\,\frac{\omega(\phi(\xi))^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}} \\ &\geq NV_{g,\phi,\omega}(\xi)\,|F_{\phi(\xi),n,p}(\phi(\xi))|\,\omega(\phi(\xi))^{\frac{1}{2}}.\end{aligned}$$

Now

$$\begin{aligned}\infty > \|GV_{(\phi,g)}(F_{\phi(\xi),n,p})\|_{A^\infty_\omega} &\geq \frac{|g(\xi)|}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}} \\ &\asymp \frac{|g(\xi)|}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}} = NV_{g,\phi,\omega}(\xi).\end{aligned} \tag{5.12}$$

If $f(z)=z$, the boundedness of the operator $GV_{(\phi,g)} : A^\infty_\omega \to A^\infty_\omega$ implies that

$$\|GV_{(\phi,g)}f\|_{A^\infty_\omega} = \sup_{z\in\mathbb{D}} \frac{|g(z)|}{(1+\varphi'(z))}\,\omega(z)^{\frac{1}{2}} \lesssim \|f\|_{A^\infty_\omega} < \infty. \tag{5.13}$$

Therefore, in the case of $|\phi(\xi)| \leq \rho_0$, $\xi\in\mathbb{D}$, we have

$$\frac{|g(\xi)|}{(1+\varphi'(\xi))}\,\frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}} \leq C_2\,\frac{|g(\xi)|}{(1+\varphi'(\xi))}\,\omega(\xi)^{\frac{1}{2}} < \infty,$$

where

$$C_2 = \sup_{|\phi(\xi)|\leq\rho_0} \left\{\omega(\phi(\xi))^{\frac{-1}{2}}\right\} < \infty.$$

Combining this with (5.12) shows that (1.7) holds.

Compactness. This is similar to the proof of (C) of Theorem 1.1.

5.8. *Proof of Theorem 1.2 (D)*

Let $0<q<p<\infty$ and suppose that $GV_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded. According to (5.9), the measure $\nu_{\phi,\omega,g}$ is a q-Carleson measure for A^p_ω. Thus, by Theorem 3.3, $\nu_{\phi,\omega,g}$ is a vanishing q-Carleson measure for A^p_ω. In this case, we have

$$\|GV_{(\phi,g)}f_n\|^q_{A^q_\omega} \to 0, \quad n\to\infty,$$

for any sequence $\{f_n\}\subset A^p_\omega$ converges to zero uniformly on compact subsets of $\mathbb{D}$. By Lemma 3.7 of [16], $GV_{(\phi,g)}$ is compact.

Next we show that (a) and (c) are equivalent. Suppose first that (c) holds. Then

$$\begin{aligned}\int_{\mathbb{D}} G_q(v_{\phi,\omega,q})(z)^{p/(p-q)} dA(z) &= \int_{\mathbb{D}} \left(\tau(z)^{2(1-\frac{q}{p})} G_q(v_{\phi,\omega,q})(z)\right)^{p/(p-q)} d\lambda(z) \\ &\asymp \int_{\mathbb{D}} GB^{\phi}_{0,p,q}(g)^{p/(p-q)} d\lambda(z) < \infty.\end{aligned} \tag{5.14}$$

Thus, according to Theorem C, $\nu_{\phi,\omega,q}$ is a q-Carleson measure for A^p_ω. Then, by (1.15),

$$\|GV_{(\phi,g)} f_n\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} |f(z)|^q\, \omega(z)^{q/2}\, d\nu_{\phi,\omega,g}(z) \lesssim \|f\|^q_{A^p_\omega},$$

for any function $f \in A^p_\omega$.

Conversely, suppose the operator $GV_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded. Then, for each function $f \in A^p_\omega$, by (1.15),

$$\|GV_{(\phi,g)}\, f\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} |f(z)|^q\, \omega(z)^{q/2}\, d\nu_{\phi,\omega,g}(z).$$

Thus, the measure $\nu_{\phi,\omega,g}$ is a q-Carleson measure for A^p_ω. According to Theorem C, $\nu_{\phi,\omega,g}$ belongs to $L^{p/(p-q)}(\mathbb{D}, dA)$. Combining this with (5.14) yields that $GB^{\phi}_{0,p,q}(g) \in L^{p/(p-q)}(\mathbb{D}, d\lambda)$.

Let $0 < q < p = \infty$ and suppose that $GV_{(\phi,g)} : A^\infty_\omega \to A^q_\omega$ is bounded. Then, by (1.15),

$$\|GV_{(\phi,g)} f\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} \frac{|f(\phi(z))|^q |g(z)|^q}{(1+\varphi'(z))^q}\, \omega(z)^{\frac{q}{2}}\, dA(z) \lesssim \|f\|^q_{A^\infty_\omega},$$

and it follows from Theorem E that the measure $\nu_{\phi,\omega,g}$ is a q-Carleson measure for A^∞_ω. Thus, by Theorem 3.3, $\nu_{\phi,\omega,g}$ is a vanishing q-Carleson measure for A^∞_ω. As in the previous case, this shows the compactness of the operator $GV_{(\phi,g)}$.

It remains to prove that (1) and (3) are equivalent when $p = \infty$. Assume first that (3) holds. Then

$$\begin{aligned}\int_{\mathbb{D}} G_q(\nu_{\phi,\omega,q})(z)\, dA(z) &= \int_{\mathbb{D}} \left(\tau(z)^2 G_q(\nu_{\phi,\omega,q})(z)\right) d\lambda(z) \\ &\asymp \int_{\mathbb{D}} GB^{\phi}_{0,p,q}(g)(z)\, d\lambda(z).\end{aligned} \tag{5.15}$$

Thus, according to Theorem E, $\nu_{\phi,\omega,q}$ is a q-Carleson measure for A^∞_ω. Then for any function $f \in A^\infty_\omega$, we have

$$\|GV_{(\phi,g)}f_n\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} |f(z)|^q\,\omega(z)^{q/2}\,d\nu_{\phi,\omega,g}(z) \lesssim \|f\|^q_{A^\infty_\omega}.$$

Conversely, suppose the operator $GV_{(\phi,g)} : A^\infty_\omega \to A^q_\omega$ is bounded. Then, for any function $f \in A^\infty_\omega$, we have

$$\|GV_{(\phi,g)}\,f\|^q_{A_\omega} \asymp \int_{\mathbb{D}} |f(z)|^q\,\omega(z)^{q/2}\,d\nu_{\phi,\omega,g}(z).$$

By assumption, this implies that the measure $\nu_{\phi,\omega,g}$ is a q-Carleson measure for A^∞_ω. According to Theorem E, $\nu_{\phi,\omega,g}$ belongs to $L^1(\mathbb{D}, dA)$. Combining this with (5.15) implies that $GB^\phi_{0,p,q}(g) \in L^1(\mathbb{D}, d\lambda)$.

6. Proofs of Proposition 1.3 and Corollary 1.5

6.1. Proof of Proposition 1.3 (A)

Suppose that the operator $GI_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded. Let $\xi \in \mathbb{D}$ be such that $|\phi(\xi)| > \rho_0$. Using the test function of Lemma E, (2.4) and (1.15), we get

$$\begin{aligned}\|F_{\phi(\xi),n,p}\|^q_{A^p_\omega} &\gtrsim \|GI_{(\phi,g)}F_{\phi(\xi),n,p}\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} \frac{|F'_{\phi(\xi),n,p}(\phi(z))|^q}{(1+\varphi'(z))^q}\,|g(z)|^q\,\omega(z)^{\frac{q}{2}}\,dA(z)\\ &\gtrsim \tau(\xi)^2\frac{|F'_{\phi(\xi),n,p}(\phi(\xi))|^q}{(1+\varphi'(\xi))^q}\,|g(\xi)|^q\,\omega(\xi)^{\frac{q}{2}},\end{aligned}$$

while Lemma 2.3 implies that

$$\|F_{\phi(\xi),n,p}\|^q_{A^p_\omega} \gtrsim \tau(\xi)^2|g(\xi)|^q\,\frac{(1+\varphi'(\phi(\xi)))^q}{(1+\varphi'(\xi))^q}\,\frac{\omega(\xi)^{\frac{q}{2}}}{\omega(\phi(\xi)^{\frac{q}{2}}}.$$

By Lemma E, we have

$$1 \gtrsim |g(\xi)|\,\frac{\tau(\xi)^{2/q}}{\tau(\phi(\xi))^{2/p}}\,\frac{1+\varphi'(\phi(\xi))}{1+\varphi'(\xi)}\,\frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}}. \tag{6.1}$$

When $|\phi(\xi)| \le \rho_0$, we have

$$\sup_{\phi(\xi)\le\rho_0} |g(\xi)|\,\frac{\tau(\xi)^{2/q}}{\tau(\phi(\xi))^{2/p}}\,\frac{1+\varphi'(\phi(\xi))}{1+\varphi'(\xi)}\,\frac{\omega(\xi)^{\frac{1}{2}}}{\omega(\phi(\xi))^{\frac{1}{2}}} < \infty.$$

Thus, (1.8) holds.

Suppose next that the operator $GI_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is compact. Let $\xi \in \mathbb{D}$ be such that $|\phi(\xi)| > \rho_0$ and define

$$f_{\phi(\xi),n,p} = \frac{F_{\phi(\xi),n,p}}{\tau(\phi(\xi))^{2/p}}, \quad \text{for } |\phi(\xi)| > \rho_0,$$

which belongs to A^p_ω and converges uniformly to zero on compact subsets of $\mathbb{D}$ as $|\phi(\xi)| \to 1$. By (2.4) and Lemma 2.3, we get

$$\begin{aligned}\|GI_{(\phi,g)} f_{\phi(\xi),n,p}\|^q_{A^q_\omega} &\asymp \int_{\mathbb{D}} \frac{|f'_{\phi(\xi),n,p}(\phi(z))|^q}{(1+\varphi'(z))^q} |g(z)|^q \omega(z)^{\frac{q}{2}} dA(z) \\ &\gtrsim \tau(\xi)^2 \frac{|f'_{\phi(\xi),n,p}(\phi(\xi))|^q}{(1+\varphi'(\xi))^q} |g(\xi)|^q \omega(\xi)^{\frac{q}{2}} \\ &\gtrsim |g(\xi)|^q \frac{\tau(\xi)^2}{\tau(\phi(\xi))^{2q/p}} \frac{(1+\varphi'(\phi(\xi)))^q}{(1+\varphi'(\xi))^q} \frac{\omega(\xi)^{\frac{q}{2}}}{\omega(\phi(\xi))^{\frac{q}{2}}}.\end{aligned}$$

Using the compactness of the operator $GI_{(\phi,g)}$, we have the desired conclusion and the proof is complete.

6.2. Proof of Proposition 1.3 (B)

Suppose that $GV_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded. By Theorem 1.2 (A), this is equivalent to $GB^\phi_{0,p,q}(g) \in L^\infty(\mathbb{D}, dA)$. By (2.4) and (2.10), we have

$$\begin{aligned}GB^\phi_{0,p,q}(g)(\phi(z)) &= \int_{\mathbb{D}} |k_{p,\phi(z)}(\phi(\xi))|^q \frac{|g(\xi)|^q}{(1+\varphi'(\xi))^q} \omega(\xi)^{q/2} dA(\xi) \\ &\geq \int_{D_\delta(z)} |k_{p,\phi(z)}(\phi(\xi))|^q \frac{|g(\xi)|^q}{(1+\varphi'(\xi))^q} \omega(\xi)^{q/2} dA(\xi) \\ &\gtrsim \tau(z)^2 |k_{p,\phi(z)}(\phi(z))|^q \frac{|g(z)|^q}{(1+\varphi'(z))^q} \omega(z)^{q/2} \\ &\gtrsim \frac{\tau(z)^2}{\tau(\phi(z))^{2q/p}} \frac{\omega(z)^{q/2}}{\omega(\phi(z))^{q/2}} \frac{|g(z)|^q}{(1+\varphi'(z))^q},\end{aligned} \tag{6.2}$$

which proves that (1.9) holds. If $GV_{(\phi,g)}$ is compact, then it follows from Theorem 1.2 (A) that $GB^\phi_{0,p,q}(g)(\phi(z)) \to 0$ as $|z| \to 1$, which completes the proof.

6.3. Proof of Corollary 1.5

(A) Let $p < q$ and suppose that $GI_{(id,g)}$ is bounded. By (2.10) and Lemma A, we have

$$\begin{aligned}|g(z)|^q &\asymp \tau(z)^{2q/p} |g(z)|^q |k_{p,z}(z)|^q \omega(z)^{\frac{q}{2}} \\ &\lesssim \frac{\tau(z)^{2q/p}}{\tau(z)^2} \int_{D_\delta(z)} |g(s)|^q |k_{p,z}(s)|^q \omega(s)^{\frac{q}{2}} dA(s) \lesssim \frac{\tau(z)^{2q/p}}{\tau(z)^2} GB^{id}_{1,p,q}(g)(z).\end{aligned}$$

Then, using the boundedness of $GI_{(id,g)}$, we obtain

$$\sup_{z\in\mathbb{D}} |g(z)|^q\, \tau(z)^{2(1-q/p)} \lesssim \sup_{z\in\mathbb{D}} GB^{id}_{1,p,q}(g)(z) < \infty.$$

Since $\tau(z)^{2(1-q/p)} \to \infty$, as $|z| \to 1$, the function g must be zero.

6.4. Proof of Corollary 1.5

(B) Let $q < p$. Using

$$\|GI_{(\phi,g)} f\|^q_{A^q_\omega} \asymp \int_{\mathbb{D}} \frac{|f'(\phi(z))|^q |g(z)|^q}{(1+\varphi'(z))^q}\, \omega(z)^{\frac{q}{2}}\, dA(z) = \|f'\|^q_{L^q(\mu_{\phi,\omega,g})} \tag{6.3}$$

(see (1.15)) and Lemma 4.2, we get $GI_{(\phi,g)} : A^p_\omega \to A^q_\omega$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S^p_\omega \to L^q(\mu_{\phi,\omega,g})$ is bounded if and only if $I_{\mu_{\phi,\omega,g}} : S^p_\omega \to L^q(\mu_{\phi,\omega,g})$ is compact if and only if the function

$$F_{\delta,\mu_{\phi,\omega,g}}(\varphi)(z) := \frac{1}{\tau(z)^2} \int_{D_\delta(z)} (1+\varphi'(\xi))^q \omega(\xi)^{-q/2}\, d\mu_{\phi,\omega,g}(\xi) \tag{6.4}$$

belongs to $L^{p/(p-q)}(\mathbb{D}, dA)$. Since $\phi = id$, we have

$$d\mu_{\phi,\omega,g}(z) = \frac{|g(z)|^q}{(1+\varphi'(z))^q}\, \omega(z)^{q/2}\, dA(z)$$

and invoking this in the condition (6.4), it becomes exactly

$$\frac{1}{\tau(z)^2} \int_{D_\delta(z)} |g(\xi)|^q\, dA(\xi) \in L^{p/(p-q)}(\mathbb{D}, dA).$$

Applying Lemma A, we get that $g \in L^r(\mathbb{D}, dA)$, with $r = pq/(p-q)$.

Conversely, suppose that $g \in L^r(\mathbb{D}, dA)$. By Hölder's inequality and (1.15), we obtain

$$\begin{aligned} \|GI_{(id,g)} f\|^q_{A^q_\omega} &\asymp \int_{\mathbb{D}} \frac{|f'(z)|^q |g(z)|^q}{(1+\varphi'(z))^q}\, \omega(z)^{\frac{q}{2}}\, dA(z) \\ &\lesssim \left(\int_{\mathbb{D}} \frac{|f'(z)|^p \omega(z)^{\frac{p}{2}}}{(1+\varphi'(z))^p}\, dA(z) \right)^{\frac{q}{p}} \left(\int_{\mathbb{D}} |g(z)|^r\, dA(z) \right)^{\frac{q}{r}} \\ &\asymp \|f\|^q_{A^p_\omega} \|g\|^q_{L^r(\mathbb{D},dA)} \lesssim \|f\|^q_{A^p_\omega}, \end{aligned} \tag{6.5}$$

which proves boundedness and completes the proof.

6.5. Proof of Corollary 1.5 (C)

Let $0 < p \le q \le \infty$. We characterize boundedness using Theorem 1.2. Suppose that $GB^{id}_{0,p,q}(g') \in L^\infty(\mathbb{D}, dA)$. It follows from (6.2) (changing g by g' and $\phi = id$),

$$\begin{aligned} GB^{id}_{0,p,q}(g')(z) &= \int_{\mathbb{D}} |k_{p,z}(\xi)|^q \, \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q} \, \omega(\xi)^{q/2} \, dA(\xi) \\ &\gtrsim \frac{\tau(z)^2}{\tau(z)^{2q/p}} \, \frac{|g'(z)|^q}{(1+\varphi'(z))^q} \asymp \left(\frac{|g'(z)|}{(1+\varphi'(z))} \, \Delta\varphi(z)^{\frac{1}{p}-\frac{1}{q}} \right)^q . \end{aligned} \tag{6.6}$$

Thus,

$$\frac{|g'(z)|}{(1+\varphi'(z))} \Delta\varphi(z)^{\frac{1}{p}-\frac{1}{q}} \in L^\infty(\mathbb{D}, dA).$$

Conversely, suppose that

$$T(g,\varphi)(z) := \frac{|g'(z)|}{(1+\varphi'(z))} \, \Delta\varphi(z)^{\frac{1}{p}-\frac{1}{q}} \in L^\infty(\mathbb{D}, dA).$$

By (2.9), we have

$$\begin{aligned} GB^{id}_{0,p,q}(g')(z) &= \int_{\mathbb{D}} |k_{p,z}(\xi)|^q \, \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q} \, \omega(\xi)^{q/2} \, dA(\xi) \\ &\lesssim \left(\tau(z)^{2(1-q/p)} \int_{\mathbb{D}} |k_{q,z}(\xi)|^q \, \Delta\varphi(z)^{1-\frac{q}{p}} \, \omega(\xi)^{q/2} \, dA(\xi) \right) \sup_{z\in\mathbb{D}} (T(g,\varphi)(z))^q . \end{aligned}$$

Since $\Delta\varphi(z) \asymp \tau(z)^{-2}$,

$$\begin{aligned} GB^{id}_{0,p,q}(g')(z) &\lesssim \left(\int_{\mathbb{D}} |k_{q,z}(\xi)|^q \, \omega(\xi)^{q/2} \, dA(\xi) \right) \sup_{z\in\mathbb{D}} (T(g,\varphi)(z))^q \\ &= \|k_{q,z}\|^q_{A^q_\omega} \sup_{z\in\mathbb{D}} (T(g,\varphi)(z))^q = \sup_{z\in\mathbb{D}} (T(g,\varphi)(z))^q . \end{aligned} \tag{6.7}$$

This finishes the proof of boundedness.

The characterization for compactness follows from Theorem 1.2, (6.7), and (6.6).

6.6. Proof of Corollary 1.5

(D) Let $0 < q < p < \infty$. We first suppose that $V_g : A^p_\omega \to A^q_\omega$ is bounded, that is, $GB^{id}_{0,p,q}(g') \in L^{\frac{p}{p-q}}(\mathbb{D}, d\lambda)$ (see Theorem 1.2). Then, by (2.4), we have

$$GB^{id}_{0,p,q}(g')(z) = \int_{\mathbb{D}} |k_{p,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q} \,\omega(\xi)^{q/2}\, dA(\xi)$$
$$\gtrsim \tau(z)^2\, |k_{p,z}(z)|^q \frac{|g'(z)|^q}{(1+\varphi'(z))^q}\, \omega(z)^{q/2}.$$

By Lemma 2.2, we obtain

$$GB^{id}_{0,p,q}(g')(z) \gtrsim \frac{\tau(z)^2}{\tau(z)^{2q/p}} \frac{|g'(z)|^q}{(1+\varphi'(z))^q} \asymp \left(\frac{|g'(z)|}{(1+\varphi'(z))}\, \Delta\varphi(z)^{\frac{1}{p}-\frac{1}{q}}\right)^q.$$

In this case, we extract that

$$\tau(z)^{2(\frac{q}{p}-1)} GB^{id}_{0,p,q}(g')(z) \gtrsim \left(\frac{|g'(z)|}{(1+\varphi'(z))}\right)^q.$$

By our assumption and the fact that $\tau(z)^{2q/p}$ is bounded, it follows that (1.11) holds.

Conversely, put $r = \frac{pq}{p-q}$. By Hölder's inequality, we obtain

$$\begin{aligned}
&GB^{id}_{0,p,q}(g')(z)^{p/(p-q)} \\
&= \left(\int_{\mathbb{D}} |k_{p,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q}\, \omega(\xi)^{q/2}\, dA(\xi)\right)^{p/(p-q)} \\
&\le \|K_z\|^{-r}_{A^p_\omega} \left(\int_{\mathbb{D}} |K_z(\xi)|^{\frac{r}{2}} \left(\frac{|g'(\xi)|}{1+\varphi'(\xi)}\right)^r \omega(\xi)^{\frac{r}{4}}\, dA(\xi)\right) \cdot \left(\int_{\mathbb{D}} |K_z(\xi)|^{\frac{p}{2}} \omega(\xi)^{\frac{p}{4}}\, dA(\xi)\right)^{\frac{q}{(p-q)}} \\
&= \frac{\|K_z\|^{r/2}_{A^{p/2}_\omega}}{\|K_z\|^r_{A^p_\omega}} \int_{\mathbb{D}} |K_z(\xi)|^{\frac{r}{2}} \left(\frac{|g'(\xi)|}{1+\varphi'(\xi)}\right)^r \omega(\xi)^{\frac{r}{4}}\, dA(\xi).
\end{aligned}$$

By Theorem A, $\frac{\|K_z\|^{r/2}_{A^{p/2}_\omega}}{\|K_z\|^r_{A^p_\omega}} \asymp \omega(z)^{\frac{r}{4}}\, \tau(z)^r$, and Fubini's theorem implies that

$$\int_{\mathbb{D}} GB^{id}_{0,p,q}(g')(z)^{p/(p-q)} \frac{dA(z)}{\tau(z)^2}$$
$$\lesssim \int_{\mathbb{D}} \left(\frac{|g'(\xi)|}{1+\varphi'(\xi)}\right)^r \omega(\xi)^{\frac{r}{4}} \left(\int_{\mathbb{D}} |K_\xi(z)|^{\frac{r}{2}}\, \omega(z)^{\frac{r}{4}}\, \tau(z)^{r-2}\, dA(z)\right) dA(\xi).$$

Since

$$\omega(\xi)^{\frac{r}{4}} \left(\int_{\mathbb{D}} |K_\xi(z)|^{\frac{r}{2}}\, \omega(z)^{\frac{r}{4}}\, \tau(z)^{r-2}\, dA(z)\right) \lesssim 1$$

(see Lemma D), the proof is complete.

6.7. Proof of Corollary 1.6

(I) Let $0 < p = q < \infty$. By (c) of Lemma 32 in [3],

$$\psi_\omega(r) \asymp (1+\varphi'(r))^{-1} \quad \text{for } r \in [0,1). \tag{6.8}$$

Therefore,

$$\begin{aligned} GB^{id}_{0,p,p}(g')(z) &= \int_{\mathbb{D}} |k_{p,z}(\xi)|^p \frac{|g'(\xi)|^p}{(1+\varphi'(\xi))^p} \, \omega(\xi)^{p/2} \, dA(\xi) \\ &\asymp \sup_{\xi\in\mathbb{D}} (\psi_\omega(\xi)|g'(\xi)|)^p \left(\int_{\mathbb{D}} |k_{p,z}(\xi)|^p \, \omega(\xi)^{p/2} \, dA(\xi) \right) \\ &= \sup_{\xi\in\mathbb{D}} (\psi_\omega(\xi)|g'(\xi)|)^p \, \|k_{p,z}\|^p_{A^p_\omega} = \sup_{\xi\in\mathbb{D}} (\psi_\omega(\xi)|g'(\xi)|)^p \, . \end{aligned} \tag{6.9}$$

The other assertion follows easily from (6.9).

6.8. Proof of Corollary 1.6

(II) Let $0 < p < q < \infty$. Note that the weighted Bergman space $A^p(\omega)$, defined in [15], is the same as the Bergman spaces A^p_W, with $W = \omega^{2/p}$. Moreover,

$$GB^{id}_{0,p,q}(g')(z) = \int_{D_\delta(z)} |k_{p,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q} \, \omega(\xi) \, dA(\xi),$$

and (2.10) is transformed to

$$|k_{p,z}(\zeta)|^q \, \omega(\zeta)^{q/p} \asymp \tau(z)^{-2q/p}, \qquad \zeta \in D_\delta(z), \tag{6.10}$$

where $k_{p,z}(\xi) = K_z(\xi)/\|k_{p,z}\|_{A^p(\omega)}$.

Let $s = \frac{2}{p} - \frac{2}{q}$. Then, by (6.8) and successively (2.4), (2.3) and (6.10), we get

$$\begin{aligned} \left(\|K_z\|^{2s}_{A^2(\omega)} \psi_\omega(z)|g'(z)| \right)^q &\lesssim \frac{\|K_z\|^{2qs}_{A^2(\omega)}}{\tau(z)^2 \omega(z)^{1-\frac{q}{p}}} \int_{D_\delta(z)} \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q} \, \omega(\xi)^{1-\frac{q}{p}} \, dA(\xi) \\ &\lesssim \frac{1}{\tau(z)^{2q/p}} \int_{D_\delta(z)} \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q} \, \omega(\xi)^{1-\frac{q}{p}} \, dA(\xi) \\ &\lesssim \int_{D_\delta(z)} |k_{p,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q} \, \omega(\xi) \, dA(\xi) \\ &\lesssim GB^{id}_{0,p,q}(g')(z) < \infty. \end{aligned}$$

Thus, to prove that the function g' vanishes on $\mathbb{D}$, it is enough to show that $\|K_z\|^{2s}_{A^2(\omega)}\psi_\omega(|z|)$ goes to infinity as $|z| \to 1$. Indeed, by (2.5) and (1.12), we have

$$\|K_z\|^{2s}_{A^2(\omega)}\psi_\omega(|z|) \asymp \frac{\tau(z)^{2(1-s)}}{(1-|z|)^t\omega(z)^s},$$

and so,

$$\lim_{|z|\to 1} \|K_z\|^{2s}_{A^2(\omega)}\psi_\omega(z) = \infty$$

because of Lemma 2.3 in [15].

6.9. Proof of Corollary 1.6

(III) Let $q < p$, and suppose that $GB^{id}_{0,p,q}(g') \in L^{p/(p-q)}(\mathbb{D}, d\lambda)$. Then

$$\begin{aligned} GB^{id}_{0,p,q}(g')(z) &\gtrsim \int_{D_\delta(z)} |k_{p,z}(\xi)|^q \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q}\,\omega(\xi)\,dA(\xi) \\ &\gtrsim \tau(z)^{-2q/p} \int_{D_\delta(z)} \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q}\,\omega(\xi)^{\frac{p-q}{p}}\,dA(\xi), \end{aligned}$$

and so it follows from the assumption that

$$\begin{aligned} &\int_{\mathbb{D}} \left(\tau(z)^{-2} \int_{D_\delta(z)} \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q}\,\omega(\xi)^{\frac{p-q}{p}}\,dA(\xi)\right)^{\frac{p}{p-q}} dA(z) \\ &\lesssim \int_{\mathbb{D}} \left(GB^{id}_{0,p,q}(g')(z)\right)^{\frac{p}{p-q}}\,d\lambda(z) < +\infty. \end{aligned} \tag{6.11}$$

Thus, using (1.15), we get

$$\|g\|_{A^{pq/(p-q)}(\omega)} \lesssim \int_{\mathbb{D}} \left(\tau(z)^{-2} \int_{D_\delta(z)} \frac{|g'(\xi)|^q}{(1+\varphi'(\xi))^q}\,\omega(\xi)^{\frac{p-q}{p}}\,dA(\xi)\right)^{\frac{p}{p-q}} dA(z).$$

This completes the proof of Corollary 1.6.

Declaration of competing interest

There is no competing interest.

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